Hamiltonian actions in integral Kahler and generalized complex geometry

by Timothy Edward Goldberg

HAMILTONIAN ACTIONS IN INTEGRAL KÄHLER AND GENERALIZED COMPLEX GEOMETRY

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This thesis consists of two parts. The first concerns a specialization of the basic case of Hamiltonian actions on symplectic manifolds, and the second a generalization of the basic case.

Brion proved a convexity result for the moment map image of an irreducible subvariety of a compact integral Kähler manifold preserved by the complexification of the Hamiltonian group action. Guillemin and Sjamaar generalized this result to irreducible subvarieties preserved only by a Borel subgroup. In another direction, O'Shea and Sjamaar proved a convexity result for the moment map image of the submanifold fixed by an anti-symplectic involution. Analogous to Guillemin and Sjamaar's generalization of Brion's theorem, in the first part of this thesis we generalize O'Shea and Sjamaar's result, proving a convexity theorem for the moment map image of the involution fixed set of an irreducible subvariety preserved by a Borel subgroup.

In the second part of this thesis, we develop the analogue of Sjamaar and Lerman's singular reduction of Hamiltonian symplectic manifolds in the context of Hamiltonian generalized complex manifolds. Specifically, we prove that if a compact Lie group acts on a generalized complex manifold in a Hamiltonian fashion, then the partition of the global quotient by orbit types induces a partition of the Lin–Tolman quotient into generalized complex manifolds. This result holds also for reduction of Hamiltonian generalized Kähler manifolds.

BIOGRAPHICAL SKETCH

Timothy Edward Goldberg was born on January 14, 1979 to Kenneth and Jeanne Goldberg in Roosevelt Hospital in the Hell's Kitchen neighborhood of Manhattan, in New York City, New York. At the age of one half of a year, he moved upstate with his parents to Saugerties, New York. Here he attended Mother Goose Nursery School, Cahill Elementary School, and finally Saugerties Junior/Senior High School, from which he graduated in 1997. From an early age, inspired and encouraged by his parents, he was fascinated by mathematics and science. His strong appreciation for both math and for sharing it with others was cultivated while accompanying his father to numerous math education conferences and events. His favorite high school classes were math, of course, especially those spectacularly taught by Debra Cacchillo.

In 1998, Timothy moved across the Hudson River and began undergraduate studies at Bard College in Annandale-on-Hudson, New York, with the generous support of a Distinguished Scientist Scholarship. Although originally undecided between mathematics and science, this indecision didn't last past the end of his first semester, when he first started reading the textbook *Proofs and Fundamentals* by his future advisor, Ethan Bloch. Although he enjoyed a full and varied liberal arts education, most, and by far the most enjoyable, of his college courses were in the Mathematics Department, at the time consisting of Ethan, Lauren Rose, Mark Halsey, and Bob McGrail. In his senior year he completed his Senior Project, *Combinatorial Laplacians of simplicial complexes*, a combination of geometry, topology, and algebra, under the supervision of his advisor, Ethan Bloch, and graduated with a Bachelor of Arts in Mathematics.

In 2002, Timothy moved a little bit west and began graduate studies at Cornell University in Ithaca, New York, with the generous support of a Teaching Assistant

Fellowship. In the spring of 2004, he took a Complex Analysis course with Clifford Earle, where he first gained a full appreciation of the beautiful geometry of the complex numbers, and also fell in love with the basic ideas, and especially the terminology (e.g. "charts", "atlases"), of manifolds. The following semester, he took the exceptional Differentiable Manifolds course offered by his future doctoral advisor, Reyer Sjamaar. Captivated by both the subject matter and the impressive clarity of the presentation, Timothy knew that he had found both his general research area and his doctoral advisor. This led him to Lie groups and equivariant symplectic geometry, whose destiny as his field of research was further cemented when he realized that if G is a Lie group with identity element $e \in G$, then its Lie algebra is T_eG, which are also, of course, his initials. In 2006, he earned a Master of Science in Mathematics. In 2010, he completed his doctoral work, and this thesis, under the supervision of his advisor, Reyer Sjamaar.

This work is dedicated to Kenneth, Jeanne, Rebecca, Andrew, Sofi, and Mega, for their unceasing love and support.

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I give my most heartfelt thanks to my family, both immediate and extended, for supporting me, inspiring me, and believing in me through everything. Thank you, Dad, for raising me on stories about math and baseball. Thank you, Mom, for encouraging me to pursue my passions, and for caring enough to learn to pronounce "Lie" correctly. Thank you, Rebecca, for helping me return to Ithaca. Thank you, Andrew, for keeping me company so many times, and for introducing me to *Lego Batman*. Thank you, David, Carole, Gloria, Myrna, Russ, Jan, Arielle, Jennifer L., Jennifer J., Nicole, and Sam for your good spirits and warm wishes. Thank you, Sofi, Mega, and my many other furry friends, for your affection and faithful companionship.

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TABLE OF CONTENTS

CHAPTER 1

NOTATION AND CONVENTIONS

Unless specified otherwise, all manifolds in this thesis will be assumed to be smooth and finite dimensional. If M is a manifold, we denote by $\mathsf{T}M$ and T^*M the tangent and cotangent bundles of M, respectively. For any smooth fiber bundle $F \to M$ over M, we write $\Gamma(M, F) = \Gamma(F)$ for the space of smooth sections of this bundle. The space of smooth vector fields on M is denoted by $\mathrm{Vec}(M) := \Gamma(\mathsf{T}M)$, and the space of smooth k-forms by $\Omega^k(M) := \Gamma(\bigwedge^k \mathsf{T}^*M)$, with

$$\Omega^\bigstar(M) := \bigoplus_{k\in\mathbb{Z}} \Omega^k(M).$$

The subspace of closed k-forms (i.e. those whose exterior derivative vanishes) will be written $\Omega^k_{\mathrm{cl}}(M)$. The notation $C^\infty(M)$ will denote the space of smooth, real-valued functions on M, and of course $\Omega^0(M) = C^\infty(M)$. For a vector field $X \in \mathbf{Vec}(M)$, we denote the Lie derivative of tensor fields in the direction X by $\mathcal{L}_X$, and the interior product of tensor fields by X by ι_X. If $f\colon M \to N$ is a smooth map between smooth manifolds M and N, we denote by $\mathsf{T}f$ or f_* the associated tangent, or pushforward map, $\mathsf{T}M \to \mathsf{T}N$. We use the notation f^* for both the pullback $\mathsf{T}^*N \to \mathsf{T}^*M$ of covectors and the pullback $\Omega(N) \to \Omega(M)$ of differential forms.

The symbol $\langle \cdot, \cdot \rangle$ will be used to denote metrics on vector spaces and manifolds, but also the natural pairing between a vector space V and its dual V^*:

$$\begin{aligned} V^* \times V &\to \mathbb{R} \\ (\lambda, v) &\mapsto \langle \lambda, v\rangle := \lambda(v). \end{aligned}$$

We will also use this notation for the pairing between V-valued and V^*-valued maps, whose result is then a real-valued map. Which use of this notation is intended will be made clear by the context. We denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the natural metric on the direct sum of a vector space V and its dual V^*, defined by

$$\langle\!\langle u + \alpha, v + \beta, : \rangle\!\rangle = \frac{1}{2} \left(\alpha(v) + \beta(u)\right)$$

for $u, v \in V$ and $\alpha, \beta \in V^*$, and also the corresponding metric on the direct sum $\mathsf{T}M \oplus \mathsf{T}^*M$ of the tangent and cotangent bundles of a manifold M.

We make use of the *musical notation* for the map between a vector space and its dual induced by a bilinear form. If $B \colon V \times V \to \mathbb{R}$ is a bilinear form on a real vector space V, then we will denote by $B^\flat \colon V \to V^*$ the map

$$v \mapsto \iota_v B := B(v, \cdot).$$

If B is non-degenerate, then $B^\flat$ is invertible and we denote its inverse by $B^\sharp := \left(B^\flat\right)^{-1}$. We also use the musical notation for the generalization of vector spaces, bilinear forms, and linear maps to vector bundles, sections of their second symmetric powers, and the associated bundle maps, respectively.

Let G be a Lie group acting smoothly on a smooth manifold M. For each $g \in G$, we will typically denote its corresponding diffeomorphism $M \to M$ by g also. The Lie algebra of a given Lie group will be denoted by the corresponding lower-case German letter, so that the Lie algebra of G is $\mathfrak{g}$. For a smooth left action $G \times M \to M$, $(g, p) \mapsto g \cdot p$, of G on M, there is an associated map $\mathfrak{g} \to \mathbf{Vec}(M)$, under which the output of $\xi \in \mathfrak{g}$ is the fundamental vector field on M associated to ξ, denoted by ξ_M and defined by

$$\xi_M\big|_p := \frac{d}{dt} \left(\exp t\xi\right) \cdot p \bigg|_{t=0}$$

for $p \in M$. This vector field is complete, and its flow is

$$\begin{aligned} M \times \mathbb{R} &\to M \\ (p, t) &\mapsto (\exp t\xi) \cdot p. \end{aligned}$$

Unless noted otherwise, all group actions in this thesis will be assumed to be left actions. We denote the adjoint representation of G on $\mathfrak{g}$ by $g \mapsto \mathrm{Ad}_g$, and the coadjoint representation of G on $\mathfrak{g}^*$ by $g \mapsto \mathrm{Coad}_g$, recalling that

$$\langle \mathrm{Coad}_g(\lambda), \xi \rangle = \langle \lambda, \mathrm{Ad}_{g^{-1}}(\xi) \rangle$$

for all $g \in G$, $\xi \in \mathfrak{g}$, and $\lambda \in \mathfrak{g}^*$.

For a compact Lie group G with maximal torus T, we will routinely embed $\mathfrak{t}^*$ in $\mathfrak{g}^*$ using the dual of the projection $\mathfrak{g} \twoheadrightarrow \mathfrak{t}$ defined by the real part of the root space decomposition:

$$\mathfrak{g} \otimes \mathbb{C} = (\mathfrak{t} \otimes \mathbb{C}) \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda,$$

where $\Lambda = \mathrm{Hom}(T, \mathbf{U}(1))$ is the weight lattice of T and $\mathfrak{g}_\lambda$ is the root space corresponding to λ,

$$\mathfrak{g}_\lambda = \{\, X \in \mathfrak{g} \otimes \mathbb{C} \mid \mathrm{Ad}_t(X) = \lambda(t) \cdot X \text{ for all } t \in T \,\}.$$

For positive $p, q, n \in \mathbb{Z}$, we let I_n be the $n \times n$ identity matrix, and denote by I_p, q the $(p+q) \times (p+q)$ matrix written in block form as

$$I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

We will have occasion to refer several classical matrix Lie groups. These are:

- the **real** and **complex general linear groups**, $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{GL}(n, \mathbb{C})$, respectively;
- the **real** and **complex special linear groups**, $\mathbf{SL}(n, \mathbb{R})$ and $\mathbf{SL}(n, \mathbb{C})$, respectively;

- the **orthogonal** and **special orthogonal linear groups** (over $\mathbb{R}$), $\mathbf{O}(n)$ and $\mathbf{SO}(n)$, respectively;
- the **unitary** and **special unitary linear groups**, $\mathbf{U}(n)$ and $\mathbf{SU}(n)$, respectively;
- the **symplectic linear group**, $\mathbf{Sp}(n)$; and
- the **indefinite orthogonal group of signature** (p, q),

$$\mathbf{O}(p,q) := \{ A \in \mathbf{GL}(p+q, \mathbb{R}) \mid A^{\mathsf{T}} I_{p,q} A = I_{p,q} \} .$$

Throughout the Introduction and Chapter 4, we use the abbreviations "GC" for "generalized complex" and "GK" for "generalized Kähler".

CHAPTER 2

INTRODUCTION

2.1 The basic setup — symplectic structures

A **symplectic manifold** is a pair (M, ω) consisting of a smooth manifold M and a **symplectic structure** ω, which is a two-form $\omega \in \Omega^2(M)$ that is:

- *closed*, i.e. $\mathrm{d}\omega = 0$, and
- *nondegenerate*, i.e. the map $\omega^\flat \colon \mathsf{T}M \to \mathsf{T}^*M$ is an isomorphism.

A symplectic structure allows us to associate to each function $H \in C^\infty(M)$ a vector field X_H, called its **Hamiltonian vector field** or **symplectic gradient**, by

$$X_H := \omega^\flat \circ \mathrm{d}H.$$

Equivalently, X_H is the unique vector field satisfying the equation

$$\mathrm{d}H = \iota_{X_H}\omega.$$

Conversely, given a vector field $X \in \mathbf{Vec}(M)$, if $X = X_H$ for some function $H \in C^\infty(M)$, then X is a **Hamiltonian vector field** and H is called its **Hamiltonian function**, (although a Hamiltonian function is unique only up to an additive constant). In a symplectic manifold's guise as a mathematical setting for classical mechanics, the assignment $H \mapsto X_H$ associates to each "energy function" on (M, ω) a vector field that describes the corresponding dynamics.

Let G be a Lie group acting smoothly on M. This action is called **symplectic** if it preserves the symplectic structure ω:

$$g^*\omega = \omega \quad \text{for all } g \in G.$$

The G-action on M is called **Hamiltonian** if it is symplectic and each ξ_M, $\xi \in \mathfrak{g}$, is a Hamiltonian vector field. In this case, there is a map $\Phi\colon M \to \mathfrak{g}^*$, called a **moment map**, satisfying the following properties.

(a) Φ is equivariant with respect to the G-action on M and the coadjoint action of G on $\mathfrak{g}^*$:

$$\Phi(g \cdot p) = \mathrm{Coad}_g\left(\Phi(p)\right) \quad \text{for all } p \in M, g \in G.$$

(b) For each $\xi \in \mathfrak{g}$, the function $\Phi^\xi \in C^\infty(M)$ defined by $p \mapsto \Phi^\xi(p) := \langle \Phi(p), \xi \rangle$ is a Hamiltonian function for the vector field ξ_M:

$$\mathrm{d}\Phi^\xi = \iota_{\xi_M}\omega.$$

The nomenclature is derived from the cases of linear and angular momentum, which can be viewed as moment maps in the sense defined above. See §22.4 of [CdS01]. A **Hamiltonian symplectic manifold** consists of a quadruple (M, ω, G, Φ) containing the data of a symplectic manifold (M, ω), a Lie group G acting symplectically on (M, ω), and a moment map $\Phi\colon M \to \mathfrak{g}^*$ for this action. If G is a compact Lie group, and $T \subset G$ and $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ are given choices of maximal torus and closed Weyl chamber, we will denote by $\Delta(M)$ the set

$$\Delta(M) := \Phi(M) \cap \mathfrak{t}^*_+.$$

Moment maps have been studied extensively, and have many remarkable properties and applications. Two of the most famous results about moment maps are the Convexity Theorem of Atiyah/Guillemin–Sternberg/Kirwan, [Ati82], [GS82], [Kir84], and the Symplectic Reduction Theorem of Marsden–Weinstein, [MW74].

Theorem 2.1.1 (Convexity Theorem). *Let (M, ω, G, Φ) be a Hamiltonian symplectic manifold, and suppose that G is compact and connected and M is compact. Let T be a maximal torus of G and $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a closed Weyl chamber. Then $\Delta(M)$ is a convex polytope.*

This theorem was first proved by Atiyah/Guillemin–Sternberg in the case that G is a torus, and later generalized to arbitrary compact and connected Lie groups by Kirwan. In the torus case, the polytope can be described as the convex hull of the moment image $\Phi(M^G)$ of the torus-fixed points M^G of M.

Theorem 2.1.2 (Symplectic Reduction Theorem). *Let (M, ω, G, Φ) be a Hamiltonian symplectic manifold, and suppose that G is compact. Let $a \in \mathfrak{g}^*$ and let $\mathcal{O}_a := \mathrm{Coad}_G(a)$ be its coadjoint orbit. If G acts freely on $\Phi^{-1}(\mathcal{O}_a)$, then the quotient space $M_a := \Phi^{-1}(\mathcal{O}_a)/G$ is a manifold, and there is a symplectic structure ω_a on M_a satisfying*

$$j^*\omega = \pi^*\omega_a,$$

where $j\colon \mu^{-1}(\mathcal{O}_a) \hookrightarrow M$ is the inclusion and $\pi\colon \mu^{-1}(\mathcal{O}_a) \twoheadrightarrow \mu^{-1}(\mathcal{O}_a)/G$ is the quotient projection.

This thesis consists of two parts, the starting point of each of which is the basic setup described above, Hamiltonian actions on symplectic manifolds, and one of the two theorems written above. First, we study a special case of the basic setup, Hamiltonian actions on integral Käher manifolds, and expanded versions of the Convexity Theorem. In the second part, we investigate a generalization of the basic setup, Hamiltonian actions on generalized complex manifolds, and generalizations of the Symplectic Reduction Theorem and related results.

Much of the material in this thesis has previously appeared in published and preprint form in [Gol09a] and [Gol09b], respectively.

2.2 A specialization of the basic setup — integral Kähler structures

A **Kähler manifold** is a symplectic manifold which is also equipped with complex and Riemannian structures which are compatible with the symplectic structure. Their study therefore lies at the intersection of symplectic, complex, and Riemannian geometry. An **integral Kähler manifold** is a Kähler manifold whose symplectic form corresponds to an integral de Rham cohomology class. Assuming also that the manifold is compact, this allows us to embed it in a complex projective space as a closed complex algebraic manifold, thus adding algebraic geometry to the mix of applicable topics. A Hamiltonian action on an integral Kähler manifold is essentially a group action which preserves all the present geometric structures and is Hamiltonian with respect to the underlying symplectic structure.

The extra structure present allows one to extend the Convexity Theorem, and describe the moment polytope more explicitly. In [Bri87], Brion proved that in this context the Convexity Theorem also applies to certain subvarieties of the integral Kähler manifold, and also described the associated moment polytope in terms of the representation theory of the group action. In [GS06], Guillemin and Sjamaar expanded Brion's result to apply to an even larger class of subvarieties of the manifold.

On the other hand, one can ask questions about "real" versions of the Convexity

Theorem, as well as other results in symplectic geometry. Here, "real" refers to **real structures** on symplectic manifolds, which are anti-symplectic involutions. For Kähler manifolds, one also asks that the involution be anti-holomorphic. In this context, a natural thing to do is to study the moment image of the real (involution-fixed) part of the manifold. This study was initiated by Duistermaat in [Dui83], and thoroughly pursued by O'Shea and Sjamaar in [OS00]. In the latter paper, the authors also proved a real version of Brion's theorem concerning the moment image of subvarieties of an integral Kähler manifold. The main theorem of the first part of this thesis, Theorem 3.1.9, is a strengthening of the real version of Brion's theorem that applies to a larger class of subvarieties, completely analogous to Guillemin and Sjamaar's strengthening of Brion's original result.

2.3 A generalization of the basic setup — generalized complex structures

The second part of this thesis is concerned with **generalized complex manifolds**. Many classical geometric structures on manifolds can be viewed as reductions of the structure group $\mathbf{GL}(n,\mathbb{R})$ of the tangent bundle $\mathsf{T}M$ of an n-dimensional manifold M, often together with various integrability conditions. These include Riemannian structures ($\mathbf{GL}(n,\mathbb{R}) \rightsquigarrow \mathbf{O}(n)$), complex structures ($\mathbf{GL}(n,\mathbb{R}) \rightsquigarrow \mathbf{GL}(n/2,\mathbb{C})$), symplectic structures ($\mathbf{GL}(n,\mathbb{R}) \rightsquigarrow \mathbf{Sp}(n)$) and Kähler structures ($\mathbf{GL}(n,\mathbb{R}) \rightsquigarrow \mathbf{U}(n/2) = \mathbf{O}(n) \cap \mathbf{GL}(n/2,\mathbb{C}) \cap \mathbf{Sp}(n)$). In generalized geometry, one instead considers reductions of the structure group of the **generalized tangent bundle**, also called the **Pontryagin bundle** or **big tangent bundle**, of M,

$$\mathbb{T}M := \mathsf{T}M \oplus \mathsf{T}^*M.$$

A priori, the structure group of $\mathbb{T}M$ is just $\mathbf{GL}(2n, \mathbb{R})$, but in fact the generalized tangent bundle has extra intrinsic structure, in the form of a natural, smoothly-varying, fiber-wise, non-degenerate, symmetric bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ of signature $(1,1)$, defined over $x \in M$ by

$$\langle\!\langle u + \alpha, v + \beta \rangle\!\rangle := \frac{1}{2} \left(\alpha(v) + \beta(u) \right)$$

for $u, v \in \mathsf{T}_x M$, $\alpha, \beta \in \mathsf{T}^*_x M$. This gives a natural reduction of the structure group of $\mathbb{T}M$ from $\mathbf{GL}(2n, \mathbb{R})$ to $\mathbf{O}(n, n)$. A **generalized geometry** is a reduction of the structure group $\mathbf{O}(n, n)$ of $\mathbb{T}M$, often together with some integrability condition. For a nice survey of several such structures, see [Vai10]. Here we will only be concerned with **generalized complex (GC) structures** and **generalized Kähler (GK) structures**, which correspond to structure group reductions

$$\mathbf{O}(n, n) \rightsquigarrow \mathbf{O}(n, n) \cap \mathbf{GL}(n, \mathbb{C}) \quad \text{and} \quad \mathbf{O}(n, n) \rightsquigarrow \mathbf{U}(n/2) \times \mathbf{U}(n/2),$$

respectively. (In order to support a GC or GK structure, it turns out that the manifold must be even-dimensional.) The integrability conditions for these structures involve the **Courant bracket** of sections of $\mathbb{T}M$, an extension of the Lie bracket of vector fields.

Generalized complex geometry was introduced by Hitchin in [Hit03], and further developed by his student Gualtieri in his doctoral thesis, [Gua03]. It serves as a common ground in which symplectic and complex geometry can meet, in the sense that either of these structures on a manifold induces a GC structure. For this reason, there has been much effort to import ideas and techniques from these other fields into the GC setting. In particular, many constructions and results from equivariant symplectic geometry have found useful analogues here. One example is that of Hamiltonian group actions and moment maps, developed in [LT06].

Lin and Tolman's construction generalizes the usual symplectic definition, and they go on to prove that one can reduce a GC manifold by its Hamiltonian symmetries, in perfect parallel to the Symplectic Reduction Theorem. Just as in the symplectic case, in order to ensure that the generalized reduced space is a manifold, one must make an assumption regarding freeness of the group action.

In [SL91], Lerman and Sjamaar proved that if the symplectic reduced space is not a manifold, then the orbit type stratification of the original symplectic manifold induces a stratification of the reduced space, each strata of which is naturally a symplectic manifold. The main result of the second part of this paper, Theorem 4.4.6, is that a completely analagous result holds if the GC reduced space is singular. We also prove that this result holds also for GK manifolds, Corollary 4.4.7.

CHAPTER 3

INTEGRAL KÄHLER MANIFOLDS

At the end of their 1982 paper, [GS82], Guillemin and Sternberg gave a description of the moment map image of an integal symplectic manifold with a Hamiltonian action of a compact group in terms of certain of the highest weights of a maximal torus of the group. In 1987, [Bri87], Brion expanded this technique and applied it to certain algebraic subvarieties of the manifold, proving a convexity theorem for their moment images, and also describing each moment polytope in terms of the **highest weight polytope**, defined in Section 3.2 below. These methods have proved very useful. They were central to Guillemin and Sjamaar's 2006 generalization of Brion's theorem, [GS06], and in proving the projective case of O'Shea and Sjamaar's theorem from 2000, [OS00, §6]. Not surprisingly, the highest weight polytope is the main tool in our present study as well. It is well-known that describing the moment polytope is often at least as difficult as proving that the moment image is a convex polytope in the first place. We are fortunate to be able to make some descriptions here.

In Section 3.1 below, we lay out some of the technical context of the results of Brion, Guillemin–Sjamaar, and O'Shea–Sjamaar mentioned above. We then proceed to describe their results in more detail, leading to a statement of the main theorem of this chapter. In Section 3.2, we describe the main tool in analyzing the moment map image in this context, the highest weight polytope. The proof of the main theorem comes in Section 3.3. Section 3.4 contains some easy but interesting corollaries to the main theorem, and Section 3.5 describes a specific example in which the main theorem can be applied.

3.1 Background

Definition 3.1.1. A **Kähler manifold** is a triple (M, ω, J) consisting of a symplectic manifold (M, ω) and an integrable complex structure $J\colon \mathsf{T}M \to \mathsf{T}M$, which are compatible in the sense that the two-tensor field g on M defined by

$$g(u, v) := \omega\left(u, J(v)\right)$$

for $x \in M$, $u, v \in \mathsf{T}_x M$, is a Riemannian metric, i.e. g is symmetric and $g(u, u) > 0$ for all nonzero $u \in \mathsf{T}M$. In particular, this implies that J preserves both ω and g. The form ω is called the **Kähler form**, and g the **Kähler metric**.

For any symplectic manifold (M, ω), because ω is a closed two-form, it represents a de Rham cohomology class of degree two, $[\omega] \in H^2_{\mathrm{dR}}(M)$. The de Rham cohomology of a manifold M contains a lattice of **integral cohomology classes**, defined to be the image of the composition

$$H(M; \mathbb{Z}) \hookrightarrow H(M; \mathbb{R}) \xrightarrow{\cong} H_{\mathrm{dR}}(M),$$

where $H(M; \mathbb{Z}) \hookrightarrow H(M; \mathbb{R})$ is the inclusion of singular cohomology corresponding to the inclusion of groups $\mathbb{Z} \hookrightarrow \mathbb{R}$, and $H(M; \mathbb{R}) \xrightarrow{\cong} H_{\mathrm{dR}}(M)$ is the de Rham isomorphism.

Definition 3.1.2. An **integral symplectic manifold** is a symplectic manifold (M, ω) such that $[\omega] \in H^2_{\mathrm{dR}}(M)$ is an integral class. An **integral Kähler manifold** is one for which the underlying symplectic structure is integral.

Throughout the remainder of this chapter, we will assume that (M, ω, J) is an integral Kähler manifold which is compact and connected. Then $[\omega]$ is the Chern class of a holomorphic line bundle L over M, and there is a Hermitian metric on L with metric connection ∇ whose curvature form satisfies

$$\operatorname{curv} \nabla = \frac{1}{2\pi i}\,\omega.$$

(See [Wel08], for instance.)

Let G be a compact and connected Lie group acting on M preserving ω and J, and suppose this is covered by an action of G on L by holomorphic, complex linear bundle automorphisms preserving the Hermitian metric. The group of all line bundle automorphisms preserving the holomorphic structure of L is a complex Lie group, so the action of G lifts to a holomorphic action of the complexification $G^{\mathbb{C}}$ of G on (M, L). The Kodaira embedding theorem implies that M can be embedded in some complex projective space as a closed complex algebraic variety, on which the action of $G^{\mathbb{C}}$ is algebraic.

Remark 3.1.3. Note that for any $p \in M$, the $G^{\mathbb{C}}$-orbit through p is the image of the algebraic map $G^{\mathbb{C}} \to M$, $g \mapsto g \cdot p$, and so by Chevalley's Theorem is a constructible set. This implies that its Zariski closure and its closure in the topology of the manifold coincide. (See Corollary 2 in §I.8 and Corollary 1 in §I.10 of [Mum99].) The same is true for an orbit of any algebraic subgroup of $G^{\mathbb{C}}$, such as a Borel subgroup. Finally, note that all complex algebraic subvarieties of M, being by definition locally closed with respect to the complex algebraic Zariski topology, are also constructible, and hence have identical closures in the manifold and Zariski topologies. Therefore, for all of these sets, there is no distinction between "closed" and "Zariski-closed".

Under these assumptions, the action of G on (M, ω) is automatically Hamiltonian, with moment map $\Phi\colon M \to \mathfrak{g}^*$ obtained as follows. Let s be any global smooth section of (M, L). Each $\xi \in \mathfrak{g}$ acts on s in two ways: Lie differentiation $\mathcal{L}_\xi$,

$$(\mathcal{L}_\xi s)(p) := \frac{d}{dt}\left[(\exp t\xi) \cdot s\left((\exp t\xi)^{-1} \cdot p\right)\right]\Big|_{t=0} \quad \text{for } p \in M,$$

and covariant differentiation $\nabla(\xi_M)$ in the direction of the fundamental vector field ξ_M on M. In [Kos70, Theorem 4.3.1], Kostant showed that their difference $\mathcal{L}_\xi - \nabla(\xi_M)$ is multiplication by an imaginary-valued function on M. Hence we can define a real linear map $\mathfrak{g} \to C^\infty(M)$, $\xi \mapsto \Phi^\xi$ by

$$\Phi^\xi = \frac{1}{2\pi i}\left(\mathcal{L}_\xi - \nabla(\xi_M)\right)$$

for each $\xi \in \mathfrak{g}$. Then Φ is defined by the equation $\langle \Phi(x), \xi \rangle = \Phi^\xi(x)$, for all $\xi \in \mathfrak{g}$ and $x \in M$. It can be shown that Φ satisfies the properties of a moment map. This description comes from [GS06, Section 2]. The assumption that the moment map here is not arbitrary, but is intimately connected to the actions of G on both M and L, is extremely important, (e.g. the proof of Lemma 3.3.2).

Suppose we have involutions $\gamma\colon G \to G$, $\tau\colon M \to M$, and $\beta\colon L \to L$ such that γ is a smooth group homomorphism, τ is anti-holomorphic and anti-symplectic, and (τ, β) is an involutive (real) bundle automorphism on (M, L) which is complex antilinear on fibers and which preserves the covariant derivative ∇ on L. Then $\gamma\colon G \to G$ induces linear involutions on $\mathfrak{g}$ and $\mathfrak{g}^*$, defined by

$$\xi \mapsto (\mathsf{T}_1\gamma)\,\xi \quad \text{for } \xi \in \mathfrak{g}$$

and

$$\lambda \mapsto (\mathsf{T}_1\gamma)^*\,\lambda = \lambda \circ (\mathsf{T}_1\gamma) \quad \text{for } \lambda \in \mathfrak{g}^*,$$

which we will also denote by γ. We further require two compatibility conditions regarding the involutions and the Hamiltonian action of G on M. We assume the properties of

(1) distribution:

$$\tau(g \cdot p) = \gamma(g) \cdot \tau(p) \quad \text{for all } g \in G,\, p \in M;$$

(2) anti-equivariance:

$$\Phi\left(\tau(p)\right) = -\gamma\left(\Phi(p)\right) \quad \text{for all } p \in M.$$

There are two obvious ways to extend $\gamma\colon \mathfrak{g} \to \mathfrak{g}$ to an involution on its complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ — holomorphically or anti-holomorphically. The anti-holomorphic is more useful for our purposes. Define $\sigma\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ by $\sigma(\xi) = \gamma(\operatorname{Re}\xi) - i\,\gamma(\operatorname{Im}\xi)$ for all $\xi \in \mathfrak{g}^{\mathbb{C}}$, where Re and Im denote the real and imaginary parts with respect to the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. The anti-holomorphic Lie algebra involution σ lifts to an anti-holomorphic Lie group involution on $G^{\mathbb{C}}$, which we will also denote by σ. In [OS00, Proposition 5.5], it is proved that under the compatibility conditions described above, the fixed set of $G^{\mathbb{C}}$ under this anti-holomorphic involution preserves the fixed set of M under τ, and this is the key property we need.

Recall that a linear involution on a vector space is diagonalizable, and has eigenvalues both or one of ± 1. Let $\mathfrak{k}$ and $\mathfrak{q}$ denote the eigenspaces of $\gamma\colon \mathfrak{g} \to \mathfrak{g}$ for $(+1)$ and (-1), respectively. We can identify $\mathfrak{k}^*$ and $\mathfrak{q}^*$ with the annihilators of $\mathfrak{q}$ and $\mathfrak{k}$, respectively, and obtain a decomposition $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{q}^*$ which is also the decomposition of $\mathfrak{g}^*$ into eigenspaces of $\gamma\colon \mathfrak{g}^* \to \mathfrak{g}^*$. From the definition of σ, we see that the $(+1)$-eigenspace of $\mathfrak{g}^{\mathbb{C}}$ under this involution is exactly $\mathfrak{k} \oplus i\mathfrak{q}$, and the (-1)-eigenspace is $\mathfrak{q} \oplus i\mathfrak{k}$.

As usual, we denote fixed sets of the actions of these involutions and these groups by superscripts. One well-known reason for requiring τ to be anti-symplectic is that the submanifold M^τ of M is then Lagrangian.

Proposition 3.1.4. *Let (M, ω) be a symplectic manifold and τ be an anti-symplectic involution on M. Then M^τ is a Lagrangian submanifold of M.*

Proof. That M^τ is a submanifold of M is well-known, in particular since τ defines an action of the compact group $\mathbb{Z}/2\mathbb{Z}$ on M, so that $M^\tau = M^{\mathbb{Z}/2\mathbb{Z}}$. To prove that M^τ is Lagrangian, it suffices to show that each tangent space of M^τ is a Lagrangian subspace of the corresponding tangent space of M. Let $x \in M$, let $\Omega := \omega_x$, and let $\tilde{\tau} := \mathsf{T}_x\tau$ be the derivative of τ at x, which is a linear involution on the tangent space T_xM. Let $\mathsf{T}_xM = V^+ \oplus V^-$ be the decomposition of V into (± 1)-eigenspaces with respect to $\tilde{\tau}$. Set $V := \mathsf{T}_x(M^\tau)$, and note that $V = (\mathsf{T}_xM)^{\tilde{\tau}} = V^+$. Since $\tau^*\omega = -\omega$, we have $\Omega(u, v) = -\Omega\left(\tilde{\tau}(u), \tilde{\tau}(v)\right)$ for all $u, v \in V$. As usual, we denote symplectic complements by a superscript Ω.

Notice that Ω vanishes on $V^\pm$, for if $u, v \in V^\pm$ then

$$\Omega(u, v) = -\Omega\left(\tilde{\tau}(u), \tilde{\tau}(v)\right) = -\Omega(\pm u, \pm v) = -\Omega(u, v),$$

so $\Omega(u, v) = 0$. Since $V = V^+$, it follows that $V \subset V^\Omega$, so V is isotropic.

Let $u, v \in V^\Omega$, and let $u = u^+ + u^-$ and $v = v^+ + v^-$ denote their decompositions according to $\mathsf{T}_xM = V^+ \oplus V^-$. Again since $V = V^+$, we know

$$\begin{aligned}
\Omega(u, v) &= \Omega(u, v^+) + \Omega(u, v^-) \\
&= \Omega(u, v^-) \\
&= \Omega(u^+, v^-) + \Omega(u^-, v^-) \\
&= \Omega(u^+, v^-).
\end{aligned}$$

By the anti-symmetry of Ω, we also have $\Omega(u,v) = \Omega(u^-, v^+)$. Therefore

$$\begin{aligned}\Omega(u,v) &= \Omega(u^+ + u^-, v^+ + v^-) \\ &= \Omega(u^+, v^+) + \Omega(u^+, v^-) + \Omega(u^-, v^+) + \Omega(u^-, v^-) \\ &= \Omega(u^+, v^-) + \Omega(u^-, v^+) \\ &= 2\,\Omega(u,v),\end{aligned}$$

so $\Omega(u,v) = 0$. Hence $V^\Omega \subset \left(V^\Omega\right)^\Omega = V$, so V is coisotropic. Thus V is Lagrangian. $\square$

Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$, and suppose it is preserved by γ. (As described in Appendix B of [OS00], such a torus can always be obtained by starting from a maximal torus of the submanifold $Q = \{\, g\,\gamma(g)^{-1} \mid g \in G \,\}$ of symmetric elements of G.) Choose a closed positive Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$. Embed $\mathfrak{t}^*$ as a vector subspace of $\mathfrak{g}^*$ in the usual way, using the real version of the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$. For any subset $A \subset M$, we let $\Delta(A) := \Phi(A) \cap \mathfrak{t}^*_+$. Notice that if $m \in M^\tau$, then $\gamma\left(\Phi(m)\right) = -\Phi(m)$, so $\Phi(m) \in \mathfrak{q}^*$. Thus $\Phi(M^\tau) \subset \mathfrak{q}^*$. The main result of [OS00] was the following essential converse. The proof required that the torus T and the positive Weyl chamber $\mathfrak{t}^*_+$ be chosen so as to be "compatible" with the involutions in a certain sense, as detailed in [OS00, Section 3].

Theorem 3.1.5. *Suppose T and $\mathfrak{t}^*_+$ are "compatible" with the involutions. Then $\Delta(M^\tau) = \Delta(M) \cap \mathfrak{q}^*$.*

Later, to the current author, Sjamaar suggested and outlined the following corollary and proof. It generalizes the result of Theorem 3.1.5, doing away with the full compatibility requirements on T and $\mathfrak{t}^*_+$.

Corollary 3.1.6 (due to Sjamaar)**.** *The equation $\Phi(M^\tau) = \Phi(M) \cap \mathfrak{q}^*$ holds.*

Therefore, Theorem 3.1.5 is true for any choice of T and $\mathfrak{t}_+^*$ such that T is γ-invariant.

Proof. In Example 2.9 of [OS00], the authors describe how if $\lambda \in \mathfrak{q}^*$, a compatible involution α on the Hamiltonian G-manifold $G \cdot \lambda$, the coadjoint orbit through λ, is given by $\alpha := -\gamma$. In Proposition 2.3 of [OS00], they prove that $(G \cdot \lambda)^\alpha = G \cdot \lambda \cap \mathfrak{q}^* = G^\gamma \cdot \lambda$.

Now let $\lambda \in \Phi(M) \cap \mathfrak{q}^*$. Since $\mathfrak{t}_+^*$ is a fundamental domain for the action of G on $\mathfrak{g}^*$, there is some $g \in G$ such that $g \cdot \lambda \in \mathfrak{t}_+^*$. Put $\lambda' = g \cdot \lambda$, and note that $\lambda \in G \cdot \lambda' \cap \mathfrak{q}^*$. By the previous paragraph, there is some $k \in G^\gamma$ such that $\lambda = k \cdot \lambda'$, so $\lambda' = k^{-1} \cdot \lambda$. Since $k^{-1} \in G^\gamma$, we have

$$\begin{aligned}
\gamma(\lambda') &= \gamma\left(k^{-1} \cdot \lambda\right) \\
&= \gamma(k^{-1}) \cdot \gamma(\lambda) \\
&= k^{-1} \cdot (-\lambda) \\
&= -(k^{-1} \cdot \lambda) \\
&= -\lambda',
\end{aligned}$$

so $\lambda' \in \mathfrak{q}^*$. Because Φ is G-equivariant, if $\lambda = \Phi(x)$, then

$$\lambda' = k^{-1} \cdot \lambda = k^{-1} \cdot \Phi(x) = \Phi(k^{-1} \cdot x),$$

so $\lambda' \in \Phi(M)$. Therefore $\lambda' \in \Phi(M) \cap \mathfrak{q}^* \cap \mathfrak{t}_+^* = \Phi(M^\tau) \cap \mathfrak{t}_+^*$. So there is some $y \in M^\tau$ with $\Phi(y) = \lambda'$, which means

$$\lambda = k \cdot \lambda' = k \cdot \Phi(y) = \Phi(k \cdot y).$$

Because $k \in G^\gamma$ and $y \in M^\tau$, we have $\tau(k \cdot y) = \gamma(k) \cdot \tau(y) = k \cdot y$, so $k \cdot y \in M^\tau$ and $\lambda \in \Phi(M^\tau)$. Thus $\Phi(M) \cap \mathfrak{q}^* \subset \Phi(M^\tau)$. The other inclusion was shown above. $\square$

Theorem 3.1.5 and Corollary 3.1.6 and their proofs do not require the presence of the line bundle or the complex structures whose existence we have assumed. By Kirwan's convexity theorem ([Kir84]), the set $\Delta(M)$ is a convex polytope in $\mathfrak{t}^*$, so $\Delta(M^\tau)$ is the intersection of a convex polytope with a linear subspace, which means it too is a convex polytope.

In the full Kähler and line bundle circumstances we have defined here, O'Shea and Sjamaar also proved the following statement, [OS00, Theorem 5.10].

Theorem 3.1.7. *Let X be a closed, irreducible, complex subvariety of M preserved by $G^{\mathbb{C}}$ and τ, and let $Y \subset M^\tau$ be the closure of any nonempty component of $X_{reg} \cap M^\tau$, where X_{reg} denotes the set of regular points in X. Then*

$$\Delta(Y) = \Delta(X) \cap \mathfrak{q}^*.$$

The main result of Brion in [Bri87] implies that $\Delta(X)$ is a convex polytope in $\mathfrak{t}^*$, so as before $\Delta(Y)$ is a convex polytope as well.

In [GS06], Sjamaar and Guillemin strengthened Brion's convexity result. Let $B \subset G^{\mathbb{C}}$ be the Borel subgroup determined by our choice $\mathfrak{t}^*_+$ of positive Weyl chamber. I.e., let $B \subset G^{\mathbb{C}}$ be generated by $\exp \mathfrak{b}$, where $\mathfrak{b} \subset \mathfrak{g}^{\mathbb{C}}$ is the sum of $\mathfrak{t} \otimes \mathbb{C}$ and the root spaces of $\mathfrak{g}^{\mathbb{C}}$ corresponding to roots which are positive with respect to our choice of $\mathfrak{t}^*_+$.

Theorem 3.1.8. *Let X be a B-invariant irreducible closed subvariety of M. Then $\Delta(X)$ is a rational convex polytope in $\mathfrak{t}^*$, the closure of the set $\mathcal{C}(X)$, (defined in Section 3.2), which is a convex polytope in the space of rational points in $\mathfrak{t}^*$.*

Here, rational means rational with respect to the weight lattice of T, embedded in a particular way in $\mathfrak{t}^*$, which we specify later. This theorem and its proof do not involve any involutions, of course.

Our main result is a combination of Theorems 3.1.7 and 3.1.8.

Theorem 3.1.9 (Main Theorem)**.** *Suppose the Borel subgroup B is preserved by the involution σ on $G^{\mathbb{C}}$. Let X be a closed, irreducible, complex subvariety of M preserved by both B and τ, and let Y be the closure of any nonempty component of $X_{reg} \cap M^{\tau}$. Then*

$$\Delta(Y) = \Delta(X) \cap \mathfrak{q}^*$$

and $\Delta(Y)$ is a rational convex polytope in $\mathfrak{t}^*$, the closure of the set $\mathcal{C}_{\gamma}(Y)$, (defined in Section 3.2), which is a convex polytope in the space of rational points in $\mathfrak{t}^*$.

Theorem 3.1.9 immediately implies the following.

Corollary 3.1.10. *Suppose B and X are as in Theorem 3.1.9. If $X^{\tau} \cap X_{reg} \neq \emptyset$, then $\Delta(X^{\tau}) = \Delta(X) \cap \mathfrak{q}^*$, and so $\Delta(X^{\tau})$ is a rational convex polytope in $\mathfrak{t}^*$.*

Notice that all of these results are specific instances of the main idea that the real part of the moment polytope is the moment polytope of the real part.

Remark 3.1.11. Given an anti-holomorphic involution on $G^{\mathbb{C}}$, the question of whether or not there exists an invariant Borel subgroup has been studied, for instance in [Ada, Section 5]. An involution σ for which the answer is "yes" is called **principal**, and these are characterized as those for whom the associated real form $(G^{\mathbb{C}})^{\sigma}$ of $G^{\mathbb{C}}$ is quasisplit. One example of this is the involution $g \mapsto \overline{g}$ on $\mathbf{SU}(n)^{\mathbb{C}} = \mathbf{SL}(n, \mathbb{C})$, where the bar denotes usual complex conjugation. Another is the involution $g \mapsto I_{n,n} \cdot \overline{g} \cdot I_{n,n}$ on $\mathbf{O}(2n, \mathbb{R})^{\mathbb{C}} = \mathbf{O}(2n, \mathbb{C})$. The corresponding real forms in these examples are $\mathbf{SL}(n, \mathbb{R})$ and $\mathbf{O}(n, n)$, respectively.

3.2 The highest weight polytope

We follow Brion's approach from [Bri87], as was done in [OS00, Section 5] and [GS06], and consider certain subsets of global holomorphic sections of (M, L) and its tensor powers. We will decompose these spaces into weight spaces under the action of T.

Let $\Lambda = \mathrm{Hom}\,(T, \mathbf{U}(1))$ be the weight lattice of T. We identify Λ with a certain lattice, also denoted Λ, in $\mathfrak{t}^*$ via the map $\lambda \in \Lambda \mapsto \frac{1}{2\pi i}\,\mathsf{T}_1\lambda \in \mathfrak{t}^*$. Here $\mathsf{T}_1\lambda$ denotes the derivative of the map λ at the identity $1 \in T$. Put $\Lambda_+ = \Lambda \cap \mathfrak{t}^*_+$, the space of dominant weights. We call a point in $\mathfrak{t}^*$ **rational** if it is contained in a rational multiple of the weight lattice. Hence the set of rational points is $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 3.2.1. The fact that γ preserves T implies that γ preserves the lattice Λ. So with respect to a basis for $\mathfrak{t}^*$ consisting of lattice elements, $\gamma|_{\mathfrak{t}^*}$ can be represented by a matrix with rational entries. Since the only eigenvalues of γ are the integers 1 and -1, we conclude that there exist bases for the eigenspaces $\mathfrak{t}^* \cap \mathfrak{k}^*$ and $\mathfrak{t}^* \cap \mathfrak{q}^*$ of $\gamma|_{\mathfrak{t}^*}$ consisting of **rational** linear combinations of lattice elements. Therefore, for each of these eigenspaces, the rational elements in the eigenspace form a dense subset of it.

Let $\Gamma_{\mathrm{hol}}(M, L)$ be the space of global holomorphic sections of (M, L), and for each $r \in \mathbb{N} = \{0, 1, 2, \ldots\}$ let $\Gamma_{\mathrm{hol}}(M, L^r)$ be the space of global holomorphic sections of the r-fold tensor product of (M, L) over $\mathbb{C}$,

$$(M, L^r) = (M, \underbrace{L \otimes L \otimes \ldots \otimes L}_{r \text{ times}}).$$

Then $\Gamma_{\mathrm{hol}}(M, L^0)$ is the space of holomorphic complex-valued functions on M. Since M is compact, we know $\Gamma_{\mathrm{hol}}(M, L^0) \cong \mathbb{C}$. Since T acts on (M, L) and by extension on each (M, L^r) by complex bundle automorphisms, T acts on the spaces of holomorphic global sections of these bundles and in particular on the smooth sections: for any such section s and any $t \in T$, the action of t on s is defined by

$$(t \cdot s)(x) := t \cdot \left[s\left(t^{-1} \cdot x\right)\right]$$

for $x \in M$.

Each $\Gamma_{\text{hol}}(M, L^r)$ decomposes under the action of T into weight spaces:

$$\Gamma_{\text{hol}}(M, L^r) = \bigoplus_{\lambda \in \Lambda} \Gamma_{\text{hol}}(M, L^r)_\lambda.$$

Let $S = \bigoplus_{r \in \mathbb{N}} \Gamma_{\text{hol}}(M, L^r)$, and for each $r \in \mathbb{N}$ put $S_{\lambda,r} = \Gamma_{\text{hol}}(M, L^r)_\lambda$. Then S has a grading by $\Lambda \times \mathbb{N}$,

$$S = \bigoplus_{(\lambda,r) \in \Lambda \times \mathbb{N}} S_{\lambda,r}.$$

Let $N = [B, B]$ be the unipotent radical of B. Then this grading of S descends to a grading of the N-invariant elements of S,

$$S^N = \bigoplus_{(\lambda,r) \in \Lambda_+ \times \mathbb{N}} S^N_{\lambda,r}.$$

(Recall that any weight that appears in the weight decomposition of S^N must be dominant.)

For any B-invariant irreducible closed complex subvariety X of M, let $I(X)$ be the homogeneous ideal of S consisting of sections that vanish identically on X, let $I(X)^N$ denote the set of N-invariant sections that vanish identically on S, let $A(X)$ be the quotient $A(X) = S^N / I(X)^N$, and let $A(X)_{r\lambda,r} = S^N_{r\lambda,r} / I(X)$.

Definition 3.2.2. The **highest weight polytope** of X is the subset $\mathcal{C}(X)$ of $\Lambda \otimes \mathbb{Q}$ defined by

$$\mathcal{C}(X) := \{\, \lambda \in \Lambda \otimes \mathbb{Q} \mid \exists r \in \mathbb{Z}_+ \text{ such that } r\lambda \in \Lambda_+ \text{ and } A(X)_{r\lambda,r} \neq 0 \,\}.$$

As detailed in [Bri87], $\mathcal{C}(X)$ is indeed a convex polytope in the $\mathbb{Q}$-vector space $\Lambda \otimes \mathbb{Q}$. The specific main result of that paper is that, if X is preserved by all of $G^{\mathbb{C}}$, then $\Delta(X) \cap (\Lambda \otimes \mathbb{Q}) = \mathcal{C}(X)$ and $\Delta(X)$ is the closure of $\mathcal{C}(X)$ in $\mathfrak{t}^*$, so $\Delta(X)$ is a rational convex polytope. The main result of [GS06] is exactly that the same statements hold even if X is only preserved by B.

To put Definition 3.2.2 another way, an element $\lambda \in \Lambda \otimes \mathbb{Q}$ is contained in $\mathcal{C}(X)$ if and only if there exists $r \in \mathbb{Z}_+$ such that $r\lambda \in \Lambda_+$ and there is a section $s \in S^N_{r\lambda,r}$ which does not vanish identically on X. An equivalent condition is that there exists $r \in \mathbb{Z}_+$ such that $r\lambda \in \Lambda_+$ and the irreducible representation of G with highest weight $r\lambda$ is a submodule of the G-module $\Gamma_{\mathrm{hol}}(M, L^r)$, and there is an element of this submodule which does not vanish identically on X. Accordingly, for any subset Z of M, we make the following definition.

Definition 3.2.3. The **γ-highest weight set** of Z is the subset $\mathcal{C}_\gamma(Z)$ of $(\Lambda \otimes \mathbb{Q}) \cap \mathfrak{q}^*$ consisting of elements $\lambda \in \mathfrak{q}^*$ for which there exists $r \in \mathbb{Z}_+$ such that the irreducible representation of G with highest weight $r\lambda$ is a submodule of the G-module $\Gamma_{\mathrm{hol}}(M, L^r)$, and there is an element of this submodule which does not vanish identically on Z.

3.3 Proof of the main theorem

Suppose the Borel subgroup B is preserved by σ. Let X be a closed, irreducible, complex subvariety of M preserved by B and τ, and let Y be the closure of any nonempty component of $X_{\mathrm{reg}} \cap M^\tau$.

Proposition 3.3.1. *The equality $\mathcal{C}_\gamma(Y) = \mathcal{C}(X) \cap \mathfrak{q}^*$ holds.*

Proof. From the definition of $\mathcal{C}_\gamma(Y)$, the inclusion $\mathcal{C}_\gamma(Y) \subset \mathcal{C}(X) \cap \mathfrak{q}^*$ is immediate.

For the other direction, suppose $\lambda \in \mathcal{C}(X) \cap \mathfrak{q}^*$. Then there is some $r \in \mathbb{Z}_+$ such that $r\lambda \in \Lambda_+$ and a section $s \in S^N_{r\lambda,r}$ which does not vanish identically on X. Similarly to the proof of Proposition 3.1.4 above, observe that Y contains a Lagrangian submanifold of X_{reg}. By the compatibility of the complex and symplectic structures of M, this Lagrangian submanifold is a totally real submanifold, which implies that Y is Zariski-dense in X. Hence any holomorphic section that vanishes on all of Y must vanish on all of X, so s cannot vanish identically on Y. Since $\lambda \in \mathfrak{q}^*$, this means that $\lambda \in \mathcal{C}_\gamma(Y)$. $\square$

Consider the identity component of the fixed set $(G^{\mathbb{C}})^\sigma$ of $G^{\mathbb{C}}$ under the involution σ. Note that its Lie subalgebra is $\mathfrak{k} \oplus i\mathfrak{q}$. Proposition 5.5 of [OS00] states that τ is equivariant under the action of this subgroup on M, which implies that this subgroup preserves the fixed point set M^τ. Let H denote the identity component of the "real Borel subgroup", B^σ. This group has the virtue of preserving both X and M^τ, which means it also preserves Y. Its Lie algebra is $\mathfrak{b}^\sigma = (\mathfrak{k} \oplus i\mathfrak{q}) \cap \mathfrak{b}$.

Lemma 3.3.2. *The γ-highest weight polytope $\mathcal{C}_\gamma(Y)$ is the set of rational points in $\Delta(Y)$.*

Proof. Let $\lambda \in \mathcal{C}_\gamma(Y)$. By Proposition 3.3.1, this means $\lambda \in \mathcal{C}(X) \cap \mathfrak{q}^*$. Then there is $r \in \mathbb{Z}_+$ and $s \in \Gamma_{\mathrm{hol}}(M, L^r)$ such that $r\lambda \in \Lambda_+$, $s \in S^N_{r\lambda,r}$, and s does not vanish identically on X. Because Y is a closed subset of the compact space M, it is itself compact, so there is an element $y \in Y$ where the smooth function $\|s\|^2$ takes its maximum value on Y. Recall that since s does not vanish on X, it does not vanish on Y, so $\|s(y)\|^2 > 0$. Because H preserves Y, Y is a union of H-orbits. Since the fundamental vector fields induced by elements of its Lie algebra $\mathfrak{b}^\sigma$ are tangent to the H-orbit through the point at which the vector field is evaluated, we see that these vector fields must be tangent to Y. Because $\|s\|^2$ achieves a maximum in Y at y, this means $\mathcal{L}_{\xi_M}\|s\|^2(y) = 0$ for all $\xi \in \mathfrak{b}^\sigma$, where ξ_M is the fundamental vector field on M induced by ξ.

Let $\mathrm{Im}\colon \mathfrak{g}^\mathbb{C} \to \mathfrak{g}$ be projection onto the imaginary component of $\mathfrak{g}^\mathbb{C}$ with respect to the real form $\mathfrak{g}$, and let $\mathrm{pr}\colon \mathfrak{b} \to \mathfrak{g}$ be the restriction of Im to $\mathfrak{b} \subset \mathfrak{g}^\mathbb{C}$. For each $\xi \in \mathfrak{g}$ let $\Phi^\xi\colon M \to \mathbb{R}$ be the function given by the pairing of Φ with elements of $\mathfrak{g}$: $\Phi^\xi := \langle \Phi, \xi \rangle$. Then [GS06, Equation 7] states that

$$\mathcal{L}_{\xi_M}\|s\|^2 = 4\pi r \left(-\lambda(\mathrm{pr}\,\xi) + \Phi^{\mathrm{pr}\,\xi}\right) \|s\|^2$$

for all $\xi \in \mathfrak{b}$. By our reasoning in the previous paragraph, this tells us that

$$0 = \mathcal{L}_{\xi_M}\|s\|^2(y) = 4\pi r \left(-\lambda(\mathrm{pr}\,\xi) + \Phi^{\mathrm{pr}\,\xi}(y)\right) \|s(y)\|^2$$

for all $\xi \in \mathfrak{b}^\sigma$. Because $\|s(y)\|^2 > 0$, this implies that $-\lambda(\mathrm{pr}\,\xi) + \Phi(\mathrm{pr}\,\xi)(y) = 0$, and hence

$$\langle \Phi(y), \xi \rangle = \lambda(\mathrm{pr}\,\xi), \tag{3.1}$$

for all $\xi \in \mathfrak{b}^\sigma$. Recall that $\lambda \in \mathfrak{q}^*$, and because $y \in M^\tau$ we also know $\Phi(y) \in \mathfrak{q}^*$. Hence if we show that $\mathfrak{q} \subset \mathrm{pr}(\mathfrak{b}^\sigma)$, then Equation 3.1 implies that $\Phi(y) = \lambda$.

Let $\varepsilon \in \mathfrak{q}$. In [GS06, page 10], it is shown that $\mathrm{pr}\colon \mathfrak{b} \to \mathfrak{g}$ is onto. Therefore there exists some $\delta \in \mathfrak{g}$ such that $\delta + i\varepsilon \in \mathfrak{b}$. Put $\zeta = \frac{1}{2}\left(\delta + i\varepsilon + \sigma(\delta + i\varepsilon)\right)$, and note that ζ is fixed by σ. Because $\mathfrak{b}$ is preserved by σ and is a vector space, we have $\zeta \in \mathfrak{b}$. Because σ is an anti-holomorphic extension of $\gamma\colon \mathfrak{g} \to \mathfrak{g}$, we compute

$$\begin{aligned}
\zeta &= \frac{1}{2}\left(\delta + i\varepsilon + \sigma(\delta + i\varepsilon)\right) \\
&= \frac{1}{2}\left(\delta + i\varepsilon + \gamma(\delta) - i\gamma(\varepsilon)\right) \\
&= \frac{1}{2}\left(\delta + \gamma(\delta)\right) + \frac{i}{2}\left(\varepsilon - (-\varepsilon)\right) \\
&= \frac{1}{2}\left(\delta + \gamma(\delta)\right) + i\varepsilon.
\end{aligned}$$

Hence $\mathrm{Im}(\zeta) = \varepsilon$, and therefore $\mathfrak{q} \subset \mathrm{pr}(\mathfrak{b}^\sigma)$, and so $\Phi(y) = \lambda$. Thus $\mathcal{C}_\gamma(Y)$ is a subset of the rational points in $\Delta(Y)$.

Now let $\lambda = \Phi(y) \in \Delta(Y)$ be a rational point. Since $Y \subset X$, we know $\Delta(Y) \subset \Delta(X)$, so λ is a rational point of $\Delta(X)$ also. By Theorem 3.1.8, this means that $\lambda \in \mathcal{C}(X)$. Since $y \in Y \subset M^\tau$ we have $\lambda = \Phi(y) \in \mathfrak{q}^*$. By Proposition 3.3.1, $\lambda \in \mathcal{C}(X) \cap \mathfrak{q}^* = \mathcal{C}_\gamma(Y)$. Thus the rational points of $\Delta(Y)$ are contained in $\mathcal{C}_\gamma(Y)$. $\square$

We can now prove our main result, Theorem 3.1.9.

Proof of Theorem 3.1.9. We know that $\Delta(Y) \subset \Delta(X) \cap \mathfrak{q}^*$. In the course of proving the main result of [GS06], Guillemin and Sjamaar proved that $\Delta(X) = \overline{\mathcal{C}(X)}$, where the bar denotes the closure. Hence $\Delta(X) \cap \mathfrak{q}^* = \overline{\mathcal{C}(X)} \cap \mathfrak{q}^*$. Because $\mathfrak{q}^*$ is equal to the closure of its rational points, as noted in Remark 3.2.1, we know that $\overline{\mathcal{C}(X)} \cap \mathfrak{q}^* = \overline{\mathcal{C}(X) \cap \mathfrak{q}^*}$. Finally, Proposition 3.3.1 implies that $\overline{\mathcal{C}(X) \cap \mathfrak{q}^*} = \overline{\mathcal{C}_\gamma(Y)}$. Therefore

$$\Delta(X) \cap \mathfrak{q}^* = \overline{\mathcal{C}_\gamma(Y)}. \tag{3.2}$$

Because Y is a closed subset of the compact space M, it is compact. So $\Phi(Y)$ is compact in $\mathfrak{g}^*$ and hence closed. Therefore its intersection with the closed positive

Weyl chamber, $\Delta(Y) = \Phi(Y) \cap \mathfrak{t}^*_+$, is also closed. By Theorem 3.3.2 we know that $\mathcal{C}_\gamma(Y) \subset \Delta(Y)$, so $\overline{\mathcal{C}_\gamma(Y)} \subset \Delta(Y)$. Putting this together with Equation 3.2, we see that $\Delta(X) \cap \mathfrak{q}^* \subset \Delta(Y)$. Thus $\Delta(X) \cap \mathfrak{q}^* = \Delta(Y)$. $\square$

3.4 Closures of Borel orbits

Throughout this section, we will assume that the Borel subgroup B is preserved by the anti-holomorphic involution σ.

The simplest example of a closed irreducible complex subvariety of M preserved by $G^{\mathbb{C}}$ is the closure of a $G^{\mathbb{C}}$ orbit: $\overline{G^{\mathbb{C}}p}$, for some $p \in M^\tau$. In [OS00, Proposition 5.5] it was shown that the "real" part of this subvariety, $(\overline{G^{\mathbb{C}}p})^\tau$, has a nice decomposition. The simplest example of a closed irreducible complex subvariety of M preserved by B is the closure of a Borel orbit, and the "real" part of this subvariety has a corresponding decomposition. The proof is the same, after intersecting everything with B.

Lemma 3.4.1. *Let H denote the identity component of the real Lie group B^σ. For every $p \in M^\tau$, the set $(Bp)^\tau$ has a finite number of components, each of which consists of a single H-orbit.*

Therefore for any $p \in M$, $\overline{Hp}$ is the closure of a component of $(\overline{Bp})_{\mathrm{reg}} \cap M^\tau$, so Theorem 3.1.9 tells us that $\Delta((\overline{Bp})^\tau) = \Delta(\overline{Bp}) \cap \mathfrak{q}^* = \Delta(\overline{Hp})$.

Because our main result is so similar to that of Theorem 3.1.8, several of the corollaries of that theorem in [GS06] lead immediately to corresponding corollaries in our situation.

Corollary 3.4.2. *Suppose B and X are as in Theorem 3.1.9, and that $X^\tau \cap X_{reg} \neq \emptyset$. Then the set of $x \in X$ such that $\Delta(X^\tau) = \Delta(\overline{Hx})$ is nonempty and Zariski-open in X. Here H is the identity component of B^σ.*

Proof. [GS06, Corollary 2.5] states that the set of $x \in X$ such that $\Delta(X) = \Delta(\overline{Bx})$ is nonempty and Zariski-dense in X. By Theorem 3.1.8 this is equivalent to the statement that $\mathcal{C}(X) = \mathcal{C}(\overline{Bx})$, which in turn implies that $\mathcal{C}(X) \cap \mathfrak{q}^* = \mathcal{C}(\overline{Bx}) \cap \mathfrak{q}^*$. By Theorem 3.1.9 and Corollary 3.1.10, this means that $\Delta(X^\tau) = \Delta\left((\overline{Bx})^\tau\right) = \Delta(\overline{Hx})$. $\square$

Corollary 3.4.3. *The collection of polytopes $\Delta(X^\tau)$, where X ranges over all B and τ-invariant irreducible closed complex subvarieties of M, is finite.*

Proof. In [GS06, Corollary 2.6] it is proved that the collection of polytopes $\Delta(X)$, where X ranges over the same set described in the statement of this corollary, is finite. Our corollary then follows immediately from the fact that each $\Delta(X^\tau) = \Delta(X) \cap \mathfrak{q}^*$, by Theorem 3.1.9. $\square$

Because $G^{\mathbb{C}}$-invariance implies B-invariance, and because M is itself both B and $G^{\mathbb{C}}$-invariant, Corollary 3.4.2 leads to the following result.

Corollary 3.4.4. *Suppose M^τ contains a regular point. Then the set of $p \in M$ for which $\Delta(\overline{Hp}) = \Delta(\overline{G'p}) = \Delta(M^\tau)$ is nonempty and Zariski-open in M. Here H is the identity component of B^σ and G' is the identity component of $(G^{\mathbb{C}})^\sigma$.*

3.5 Examples

Probably the most abundant source of examples to which the theorems in this thesis apply is the constructions in the proof of the Borel–Weil Theorem. Suppose G is a compact and connected Lie group, $T \subset G$ is a maximal torus, $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ is a choice of positive Weyl chamber, and B is the Borel subgroup of $G^{\mathbb{C}}$ corresponding to $\mathfrak{t}^*_+$. Then for each choice of dominant weight $\lambda \in \mathfrak{t}^*_+$, we can construct an integral, compact, and connected Kähler manifold in the form of a complex flag variety $M_\lambda := G^{\mathbb{C}}/P_\lambda$, and a holomorphic line bundle $L_{-\lambda}$ over M whose Chern class is represented by M_λ's Kähler form. Here P_λ is the parabolic subgroup of $G^{\mathbb{C}}$ corresponding to λ. Furthermore, the group G acts on $(M_\lambda, L_{-\lambda})$ in a natural Kählerian fashion. A thorough treatment of this material can be found in Section 4.12 of [DK00].

For any choice of Lie group involution γ on G, so long as σ preserves the Borel subgroup B and the parabolic subgroup P_λ, we can easily construct involutions τ on M_λ and β on $L_{-\lambda}$ so that all of requirements described in Section 3.1 are satisfied. As in Section 3.4, let $x \in (M_\lambda)^\tau$, let $X = \overline{Bx}$, and let $Y = \overline{Hx}$, where H is the identity component of B^σ. So long as $(M_\lambda)^\tau$ is nonempty, we have a situation where we can apply all of the results of this paper. For an added twist, we can let $G = U \times U$ be the product of compact Lie groups. By pre-composing the action of G on M_λ with the diagonal map $U \to U \times U$, $u \mapsto (u, u)$, we obtain an action of U on M_λ. It is well-known that this action is Hamiltonian with moment map obtained by post-composing the G-moment map with the projection $(\mathfrak{u} \oplus \mathfrak{u})^* \cong \mathfrak{u}^* \oplus \mathfrak{u}^* \to \mathfrak{u}^*$ defined by dualizing the diagonal map $\mathfrak{u} \to \mathfrak{u} \oplus \mathfrak{u}$.

For a specific example, let $U = \mathbf{SU}(2)$, so that $U^{\mathbb{C}} = \mathbf{SL}(2, \mathbb{C})$. Let T consist of the diagonal matrices in U, and B the upper triangular matrices in $U^{\mathbb{C}}$. Then

$N = [B, B]$ consists of upper triangular matrices in $U^{\mathbb{C}}$ with 1's along the diagonal. Define $\alpha \in \mathfrak{t}^*$ by

$$\begin{pmatrix} 2\pi x i & 0 \\ 0 & -2\pi x i \end{pmatrix} \mapsto x \in \mathbb{R}.$$

Then $\mathfrak{t}^* \cong \mathbb{R}\cdot\alpha \subset \mathfrak{g}^*$, the positive Weyl chamber corresponding to B is $\mathfrak{t}^*_+ = \mathbb{R}_{\geq 0}\cdot\alpha$, and the weight lattice of (U, T) is $\Lambda = \mathbb{Z} \cdot \alpha$, so $\Lambda_+ = \mathbb{N} \cdot \alpha$.

Now let $G = U \times U$, and take $T \times T$ as a maximal torus and $B \times B$ as a Borel subgroup of $G^{\mathbb{C}} = U^{\mathbb{C}} \times U^{\mathbb{C}}$. The closed positive Weyl chamber of $\mathfrak{t}^* \oplus \mathfrak{t}^*$ is then $\mathfrak{t}^*_+ \times \mathfrak{t}^*_+$, the weight lattice is $\Lambda \times \Lambda$, and the set of dominant weights is $\Lambda_+ \times \Lambda_+$. Let $\lambda_1, \lambda_2 \in \Lambda_+$ be nonzero dominant weights of (U, T). Then $(\lambda_1, \lambda_2) \in \Lambda_+ \times \Lambda_+$ is a nonzero dominant weight of $(G, T \times T)$, and the corresponding parabolic subgroup is $B \times B$. Our flag variety in this case is

$$M := G^{\mathbb{C}}/(B \times B) = (U^{\mathbb{C}} \times U^{\mathbb{C}})/(B \times B) \cong (U^{\mathbb{C}}/B) \times (U^{\mathbb{C}}/B).$$

Consider the map $U^{\mathbb{C}} \to \mathbb{CP}^1$, where $\mathbb{CP}^1$ is one-dimensional complex projective space, defined by projection onto the first column:

$$\begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix} \mapsto [x_0 : x_1]. \tag{3.3}$$

Note that this map is invariant under right multiplication by B, since

$$\begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix} \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} ax_0 & cx_0 + a^{-1}y_0 \\ ax_1 & cx_1 + a^{-1}y_1 \end{pmatrix} \mapsto [ax_0 : ax_1] = [x_0 : x_1]$$

for all $\begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix} \in U^{\mathbb{C}}$, $\begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \in B$. Hence (3.3) induces a map $U^{\mathbb{C}}/B \to \mathbb{CP}^1$, which is actually an isomorphism since it has inverse

$$[x_0 : x_1] \mapsto \begin{pmatrix} x_0 & 0 \\ x_1 & x_0^{-1} \end{pmatrix} \bmod B = \begin{pmatrix} x_0 & -x_1^{-1} \\ x_1 & 0 \end{pmatrix} \bmod B.$$

The above map should be interpreted as follows. If $x_1 = 0$, then use the first formula. If $x_0 = 0$, then use the second. If both of $x_0, x_1 \in \mathbb{C}$ are nonzero, then the two formulas coincide, since

$$\begin{pmatrix} x_0 & 0 \\ x_1 & x_0^{-1} \end{pmatrix} \begin{pmatrix} 1 & -x_0^{-1}x_1^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_0 & -x_1^{-1} \\ x_1 & 0 \end{pmatrix}.$$

Furthermore, (3.3) is equivariant with respect to left multiplication of $U^{\mathbb{C}}$ on itself and the standard action of $U^{\mathbb{C}} = \mathbf{SL}(2, \mathbb{C})$ on $\mathbb{CP}^1$. Since left and right multiplication within $U^{\mathbb{C}}$ commute, the isomorphism $U^{\mathbb{C}}/B \cong \mathbb{CP}^1$ is hence $U^{\mathbb{C}}$-equivariant. It follows that $M \cong \mathbb{CP}^1 \times \mathbb{CP}^1$, also $U^{\mathbb{C}}$-equivariantly.

Let $L_{(-\lambda_1,-\lambda_2)}$ denote the holomorphic line bundle over M constructed from the Borel–Weil Theorem. For involutions on U, M, and L, we can take standard complex conjugation. For $G = U \times U$ this means conjugation on each factor. It is easily verified that these satisfy all of the necessary compatibility conditions.

Let $r \in \mathbb{Z}_+$. Then $(r\lambda_1, r\lambda_2)$ is also a dominant weight, so we can repeat the above construction, but we simply have $L_{(-r\lambda_1,-r\lambda_2)} \cong \left(L_{(-\lambda_1,-\lambda_2)}\right)^r$. Note also that

$$L \cong L_{-\lambda_1} \boxtimes L_{-\lambda_2} \quad \text{and} \quad L^r \cong L_{-r\lambda_1} \boxtimes L_{-r\lambda_2}. \tag{3.4}$$

Recall that the space $\Gamma_{\mathrm{hol}}(M, L^r)$ of global holomorphic sections of (M, L^r) is isomorphic as a G-representation to $V(r\lambda_1, r\lambda_2)^*$, the dual of the irreducible representation of G with highest weight $(r\lambda_1, r\lambda_2)$. Similarly, $\Gamma_{\mathrm{hol}}(\mathbb{CP}^1, L_{-r\lambda_1}) \cong V(r\lambda_1)^*$ and $\Gamma_{\mathrm{hol}}(\mathbb{CP}^1, L_{-r\lambda_2}) \cong V(r\lambda_2)^*$. In general, we have the formulas $V(r\lambda_1)^* \cong V(-w_0 r\lambda_1)$ and $V(r\lambda_2)^* \cong V(-w_0 r\lambda_2)$, where w_0 is the longest element of the Weyl group of (U, T). For $U = \mathbf{SU}(2)$, w_0 is the identity. Together with the equalities in (3.4), this implies that

$$\Gamma_{\mathrm{hol}}(M, L^r) = \Gamma_{\mathrm{hol}}(\mathbb{CP}^1, L_{-r\lambda_1}) \otimes \Gamma_{\mathrm{hol}}(\mathbb{CP}^1, L_{-r\lambda_1}) \cong V(r\lambda_1) \otimes V(r\lambda_2).$$

If $\lambda \in \Lambda_+$ is a dominant weight of (U, T), then V_λ is equivalent to the space of homogeneous complex polynomials of degree λ in two variables. (See pages 305–306 of [DK00].) If F is such a polynomial and $u \in U^{\mathbb{C}}$, then $(u \cdot F)(x, y) = F\left(u^{-1}(x, y)\right)$, where $U^{\mathbb{C}}$ acts on $\mathbb{C}^2$ in the usual way. Using this description, we see that $\Gamma_{\text{hol}}(M, L^r) \cong V(r\lambda_1) \otimes V(r\lambda_2)$ can be viewed as the space of complex polynomials $F(x_0, x_1, y_0, y_1)$ which are homogenous of degree $r\lambda_1$ in the first two variables and homogeneous of degree $r\lambda_2$ in the last two variables, which is a vector space of dimension $(r\lambda + 1)(r\lambda_2 + 1)$.

By the Clebsch-Gordan formula, the dominant weights that appear as highest weights in the decomposition of $V(r\lambda_1) \otimes V(r\lambda_2)$ into irreducible representations are exactly

$$r(\lambda_1 + \lambda_2), r(\lambda_1 + \lambda_2) - 2, r(\lambda_1 + \lambda_2) - 4, \ldots, r\left|\lambda_1 - \lambda_2\right|,$$

which can also be written as

$$r(\lambda_1 + \lambda_2) - 2k \quad \text{for integers } k = 0, \ldots, \min\{r\lambda_1, r\lambda_2\}. \tag{3.5}$$

For each of these weights, there is a one-dimensional subspace of $V(r\lambda_1) \otimes V(r\lambda_2)$ which is N-invariant and on which T acts by the given weight. Some careful computation shows that, for each $k = 0, \ldots, \min\{r\lambda_1, r\lambda_2\}$, this one-dimensional subspace is the complex span of the polynomial

$$\begin{aligned} F_{r,k}(x_0, x_1, y_0, y_1) &:= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} x_0^j x_1^{r\lambda_1 - j} y_0^{k-j} y_1^{r\lambda_2 - k + j} \\ &= x_1^{r\lambda_1 - k} y_1^{r\lambda_2 - k} \left(x_0 y_1 - y_0 x_1\right)^k. \end{aligned} \tag{3.6}$$

The two formulas for $F_{r,k}$ given in (3.6) are equal by the Binomial Expansion

Theorem:

$$
\begin{aligned}
x_1^{r\lambda_1-k}y_1^{r\lambda_2-k}\left(x_0y_1-y_0x_1\right)^k &= x_1^{r\lambda_1-k}y_1^{r\lambda_2-k}\left(\sum_{j=0}^{k}\binom{k}{j}(x_0y_1)^j(-y_0x_1)^{k-j}\right)\\
&= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}x_0^jy_1^{j+r\lambda_2-k}y_0^{k-j}x_1^{k-j+r\lambda_1-k}\\
&= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}x_0^jx_1^{r\lambda_1-j}y_0^{k-j}y_1^{r\lambda_2-k+j}.
\end{aligned}
$$

From the first formula, it is easy to see that multiples of $F_{r,k}$ transform under T according to the weight $r(\lambda_1+\lambda_2)-2k$, while the second formula allows easy verification that $F_{r,k}$ and its multiples are invariant under N. Indeed, if

$$
t=\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\in T
$$

then

$$
\begin{aligned}
&\left(t\cdot F_{r,k}\right)(x_0,x_1,y_0,y_1)\\
&\qquad = F_{r,k}\left(t^{-1}(x_0,x_1),t^{-1}(y_0,y_1)\right)\\
&\qquad = F_{r,k}(a^{-1}x_0,ax_1,a^{-1}y_0,ay_1)\\
&\qquad = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(a^{-1}x_0)^j(ax_1)^{r\lambda_1-j}(a^{-1}y_0)^{k-j}(ay_1)^{r\lambda_2-k+j}\\
&\qquad = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a^{-j+r\lambda_1-j-k+j+r\lambda_2-k+j}x_0^jx_1^{r\lambda_1-j}y_0^{k-j}y_1^{r\lambda_2-k+j}\\
&\qquad = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a^{r(\lambda_1+\lambda_2)-2k}x_0^jx_1^{r\lambda_1-j}y_0^{k-j}y_1^{r\lambda_2-k+j}\\
&\qquad = a^{r(\lambda_1+\lambda_2)-2k}\cdot F_{r,k}(x_0,x_1,y_0,y_1)\\
&\qquad = \left(r(\lambda_1+\lambda_2)-2k\right)(t)\cdot F_{r,k}(x_0,x_1,y_0,y_1),
\end{aligned}
$$

where here we regard the weight $r(\lambda_1+\lambda_2)-2k$ in its form as an element of

$\mathrm{Hom}\,(T, \mathbf{U}(1))$, where it is defined by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a^{r(\lambda_1+\lambda_2)-2k}.$$

Also, if

$$n = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in N$$

then

$$\begin{aligned}
(n \cdot F_{r,k})\,(x_0, x_1, y_0, y_1) &= F_{r,k}\left(n^{-1}(x_0, x_1), n^{-1}(y_0, y_1)\right) \\
&= F_{r,k}(x_0 - cx_1, x_1, y_0 - cy_1, y_1) \\
&= x_1^{r\lambda_1-k} y_1^{r\lambda_2-k}\left[(x_0 - cx_1)y_1 - (y_0 - cy_1)x_1\right]^k \\
&= x_1^{r\lambda_1-k} y_1^{r\lambda_2-k}\left(x_0y_1 - cx_1y_1 - y_0x_1 + cy_1x_1\right)^k \\
&= x_1^{r\lambda_1-k} y_1^{r\lambda_2-k}\left(x_0y_1 - y_0x_1\right)^k \\
&= F_{r,k}(x_0, x_1, y_0, y_1).
\end{aligned}$$

Fix $x = ([x_0 : x_1], [y_0 : y_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, and put $X = \overline{Bx}$. For any

$$b = \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \in B,$$

we have

$$\begin{aligned}
F_{r,k}(bx) &= F_{r,k}(ax_0 + cx_1, a^{-1}x_1, ay_0 + cy_1, a^{-1}y_1) \\
&= \left(a^{-1}x_1\right)^{r\lambda_1-k}\left(a^{-1}y_1\right)^{r\lambda_2-k}\left[(ax_0 + cx_1)(a^{-1}y_1) - (ay_0 + cy_1)(a^{-1}x_1)\right]^k \\
&= a^{2k-r\lambda_1-r\lambda_2} x_1^{r\lambda_1-k} y_1^{r\lambda_2-k}\left(x_0y_1 + a^{-1}cx_1y_1 - y_0x_1 - a^{-1}cy_1x_1\right)^k \\
&= a^{2k-r\lambda_1-r\lambda_2} x_1^{r\lambda_1-k} y_1^{r\lambda_2-k}\left(x_0y_1 - y_0x_1\right)^k \\
&= a^{2k-r\lambda_1-r\lambda_2} \cdot F_{r,k}(x).
\end{aligned}$$

Therefore $F_{r,k}(bx)$ is a nonzero multiple of $F_{r,k}(x)$ for any $b \in B$, so $F_{r,k}$ vanishes on X exactly when $F_{r,k}(x) = 0$.

The specific value of x will determine X, and there are only finitely many possibilities. Before delving into cases, note that

$$\begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \cdot [z_0 : z_1] = [az_0 + cz_1 : a^{-1}z_1].$$

for $a, c \in \mathbb{C}$, $[z_0 : z_1] \in \mathbb{CP}^1$. Note that the point $[1 : 0] \in \mathbb{CP}^1$ is fixed by B, and for any $[w_0 : 1] \in \mathbb{CP}^1$, we have

$$\begin{aligned} \begin{pmatrix} 1 & w_0 - \frac{z_0}{z_1} \\ 0 & 1 \end{pmatrix} \cdot [z_0 : z_1] &= \left[z_0 + \left(w_0 - \frac{z_0}{z_1}\right) z_1 : z_1\right] \\ &= [z_1 w_0 : z_1] \\ &= [w_0 : 1]. \end{aligned}$$

Thus $B \cdot [z_0 : z_1]$ contains all points of the form $[w_0 : 1]$ for $w_0 \in \mathbb{C}$, which is a dense subset of $\mathbb{CP}^1$. Therefore

$$\overline{B \cdot [z_0 : z_1]} = \begin{cases} \{[z_0 : z_1]\} & \text{if } z_1 = 0, \\ \mathbb{CP}^1 & \text{if } z_1 \neq 0. \end{cases} \tag{3.7}$$

Let $D \colon \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ denote the diagonal embedding. We will now determine the different possibilities for $X = \overline{Bx}$, as well as their highest weight polytopes $\mathcal{C}(X)$ and their moment polytopes $\Delta(X) = \overline{\mathcal{C}(X)}$. In the present situation, observe that

$$\mathcal{C}(X) = \left\{ \frac{r\lambda_1 + r\lambda_2 - 2k}{r} \;\middle|\; F_{r,k}(x) \neq 0 \right\}.$$

Case 1: $x \in D(\mathbb{CP}^1)$.

By (3.7), if $x = D([1:0])$ then $X = \{([1:0],[1:0])\}$. In this case, since

$$F_{r,k}(x) = 0^{r\lambda_1 - k} \cdot 0^{r\lambda_2 - k} \cdot 0^k = 0,$$

we have $\Delta(X) = \mathcal{C}(X) = \emptyset$.

If $x \neq D([1:0])$ then $X = D(\mathbb{CP}^1)$. In this case, we can write $x = ([x_0, 1], [x_0, 1])$ for some $x_0 \in \mathbb{C}$, so

$$F_{r,k}(x) = 1^{r\lambda_1 - k} \cdot 1^{r\lambda_2 - k} \cdot 0^k = 0^k.$$

Therefore $F_{r,k}(x) \neq 0$ if and only if $k = 0$, so

$$\Delta(X) = \mathcal{C}(X) = \left\{ \frac{r\lambda_1 + r\lambda_2}{r} \right\} = \{\lambda_1 + \lambda_2\}.$$

Case 2: $x = ([x_0 : x_1], [y_0 : y_1]) \notin D(\mathbb{CP}^1)$ with $x_1 y_1 = 0$.

In this case, we have exactly one of x_1, y_1 equal to zero. If $x_1 = 0$, then by (3.7) we have $X = \{[1:0]\} \times \mathbb{CP}^1$. If $y_1 = 0$, then $X = \mathbb{CP}^1 \times \{[1:0]\}$.

Suppose now that $x_1 = 0$. Then without loss of generality we can write $x = ([1, 0], [y_0, 1])$, so

$$F_{r,k}(x) = 0^{r\lambda_1 - k} \cdot 1^{r\lambda_2 - k} \cdot 1^k = 0^{r\lambda_1 - k},$$

so $F_{r,k}(x) \neq 0$ if and only if $k = r\lambda_1$. By (3.5), this number occurs as a value of k if and only if $\lambda_1 < \lambda_2$, in which case the weight is $r\lambda_1 + r\lambda_2 - 2r\lambda_1 = r\lambda_2 - r\lambda_1$. Therefore

$$\Delta(X) = \mathcal{C}(X) = \begin{cases} \emptyset & \text{if } \lambda_1 > \lambda_2, \\ \{\lambda_2 - \lambda_1\} & \text{if } \lambda_1 < \lambda_2. \end{cases}$$

By symmetry, if $y_1 = 0$ then

$$\Delta(X) = \mathcal{C}(X) = \begin{cases} \emptyset & \text{if } \lambda_1 < \lambda_2, \\ \{\lambda_2 - \lambda_1\} & \text{if } \lambda_1 > \lambda_2. \end{cases}$$

Case 3: $x = ([x_0 : x_1], [y_0 : y_1]) \notin D(\mathbb{CP}^1)$ with $x_1 y_1 \neq 0$.

By scaling the homogeneous coordinates, we can assume without loss of generality that $x = ([x_0 : 1], [y_0 : 1])$ with $x_0 \neq y_0$. For any $z_0, w_0 \in \mathbb{C}$ with $z_0 \neq w_0$, we claim that $([z_0 : 1], [w_0 : 1]) \in Bx$. Indeed, if we let a be some square root of the complex number $\frac{z_0 - w_0}{x_0 - y_0}$, and $c = \frac{x_0 w_0 - y_0 z_0}{a(x_0 - y_0)}$, then

$$\begin{aligned}
\begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \cdot [x_0, x_1] &= \left[a x_0 + c : a^{-1} \right] \\
&= \left[a^2 x_0 + ac : 1 \right] \\
&= \left[\frac{z_0 - w_0}{x_0 - y_0} \cdot x_0 + \frac{x_0 w_0 - y_0 z_0}{x_0 - y_0} : 1 \right] \\
&= \left[\frac{x_0 z_0 - x_0 w_0 + x_0 w_0 - y_0 z_0}{x_0 - y_0} : 1 \right] \\
&= \left[\frac{x_0 z_0 - y_0 z_0}{x_0 - y_0} : 1 \right] \\
&= [z_0 : 1] \, ,
\end{aligned}$$

and similarly

$$\begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \cdot [y_0, y_1] = [w_0 : 1] \, .$$

Since $\{ ([z_0 : 1], [w_0 : 1]) \mid z_0, w_0 \in \mathbb{C}, \, z_0 \neq w_0 \}$ is dense in $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, we have $X = \overline{Bx} = M$.

Continuing to write $x = ([x_0 : 1], [y_0 : 1])$ with $x_0 \neq y_0$, we have

$$F_{r,k}(x) = 1^{r\lambda_1 - k} \cdot 1^{r\lambda_2 - k} \cdot (x_0 - y_0)^k = (x_0 - y_0)^k,$$

which is never zero. Therefore

$$\begin{aligned}
\mathcal{C}(X) &= \left\{ \frac{r\lambda_1 + r\lambda_2 - 2k}{r} \;\middle|\; r \in \mathbb{Z}_+,\, k = 0, \dots, \min\{r\lambda_1, r\lambda_2\} \right\} \\
&= \left\{ \lambda_1 + \lambda_2 - \frac{2k}{r} \;\middle|\; r \in \mathbb{Z}_+,\, k = 0, \dots, \min\{r\lambda_1, r\lambda_2\} \right\} \\
&= \left\{ \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 - \frac{2}{r}, \lambda_1 + \lambda_2 - \frac{4}{r}, \dots, |\lambda_1 - \lambda_2| \;\middle|\; r \in \mathbb{Z}_+ \right\} \\
&= \{\, q \in \mathbb{Q} \mid \lambda_1 + \lambda_2 \geq q \geq |\lambda_1 - \lambda_2| \,\},
\end{aligned}$$

so $\Delta(X)$ is the closed line segment from $|\lambda_1 - \lambda_2|$ to $\lambda_1 + \lambda_2$.

The inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2$ induces an inclusion $\mathbb{RP}^1 \hookrightarrow \mathbb{CP}^1$, and elements of $\mathbb{RP}^1$ inside $\mathbb{CP}^1$ can be identified as those $[z_0 : z_1] \in \mathbb{CP}^1$ such that whichever of the ratios z_0/z_1 and z_1/z_0 are defined are real numbers. Note that $M^\tau = \mathbb{RP}^1 \times \mathbb{RP}^1 \subset \mathbb{CP}^1 \times \mathbb{CP}^1 = M$.

Suppose now that $x \in M^\tau$, so that X is preserved under τ and X^τ is nonempty. If X is M, $D(\mathbb{CP}^1)$, the product of $\mathbb{CP}^1$ and $\{[1:0]\}$, or just $\{[1:0]\} \times \{[1:0]\}$, then X^τ is $\mathbb{RP}^1 \times \mathbb{RP}^1$, $D(\mathbb{RP}^1)$, the product of $\mathbb{RP}^1$ and $\{[1:0]\}$, or just $\{[1:0]\} \times \{[1:0]\}$, respectively. Let $Y = X^\tau$. Because $\mathfrak{g} = \mathfrak{su}(2)$, all entries of elements of $\mathfrak{t}$ are pure imaginary, so γ acts on $\mathfrak{t}^*$ by negation, so $\mathfrak{t}^* \subset \mathfrak{q}^*$. Then Theorem 3.1.9 implies that $\Delta(Y) = \Delta(X) \cap \mathfrak{q}^* = \Delta(X)$, so that $\Delta(Y)$ is always the same polytope as $\Delta(X)$.

By an argument similar to that for Borel orbits in $\mathbb{CP}^1$, one can show that

$$\overline{G^{\mathbb{C}} \cdot [z_0 : z_1]} = \mathbb{CP}^1 \quad \text{for any } [z_0 : z_1] \in \mathbb{CP}^1. \tag{3.8}$$

It follows from this that if $x \in D(\mathbb{CP}^1)$, then $\overline{G^{\mathbb{C}} x} = D(\mathbb{CP}^1)$. By **Case 3** above, if $x = ([x_0 : x_1], [y_0 : y_1]) \notin D(\mathbb{CP}^1)$ with $x_1 y_1 \neq 0$, then $\overline{Bx} = M$, which means that $\overline{G^{\mathbb{C}} x} = M$ as well. In fact, these are the only possible closures of $G^{\mathbb{C}}$-orbits in

M. The remaining possibility to account for is **Case 2**, where $x = ([x_0 : x_1], [y_0 : y_1]) \notin D(\mathbb{CP}^1)$ with $x_1 y_1 = 0$. As discussed above, this means we can write x as either $x = ([1:0],[y_0,1])$ or $x = ([x_0:1],[1:0])$, depending on which of x_1 and y_1 is zero.

Let $([z_0 : 1], [w_0 : 1]) \in M$ be such that $z_0 \neq w_0$, and note that the set of all such points is dense in M. Then some easy computations show that

$$\begin{pmatrix} z_0 & \frac{w_o}{z_0 - w_0} - y_0 z_0 \\ 1 & \frac{1}{z_0 - w_0} - y_0 \end{pmatrix} \cdot ([1:0],[y_0:1]) = ([z_0:1],[w_0:1]),$$

$$\begin{pmatrix} w_0 & \frac{z_o}{w_0 - z_0} - x_0 w_0 \\ 1 & \frac{1}{w_0 - z_0} - x_0 \end{pmatrix} \cdot ([x_0:1],[1:0]) = ([z_0:1],[w_0:1]),$$

and

$$\begin{pmatrix} z_0 & \frac{w_o}{z_0 - w_0} - y_0 z_0 \\ 1 & \frac{1}{z_0 - w_0} - y_0 \end{pmatrix}, \begin{pmatrix} w_0 & \frac{z_o}{w_0 - z_0} - x_0 w_0 \\ 1 & \frac{1}{w_0 - z_0} - x_0 \end{pmatrix} \in \mathbf{SL}(2, \mathbb{C}) = G^{\mathbb{C}}.$$

Therefore, in **Case 2** we have $\overline{G^{\mathbb{C}} x} = M$.

So the only possibilities for closures of $G^{\mathbb{C}}$-orbits on M are M and $D(\mathbb{CP}^1)$, whose real parts are $\mathbb{RP}^1 \times \mathbb{RP}^1$ and $D(\mathbb{RP}^1)$, respectively. The corresponding moment polytopes are the closed line segment $[|\lambda_1 - \lambda_2|, \lambda_1 + \lambda_2]$ and the point $\{\lambda_1 + \lambda_2\}$, respectively. Note that three of the closures of Borel orbits described above are not preserved by $G^{\mathbb{C}}$, and for the Borel case we obtain an additional possible moment polytope (or two additional ones, if we include the empty set). So even in this relatively easy example, we find some situations to which the results of O'Shea and Sjamaar, such as Theorem 3.1.7 above, do not apply, while the new results of this thesis, such as Theorem 3.1.9, do.

CHAPTER 4

GENERALIZED COMPLEX MANIFOLDS

Generalized complex geometry is a rich and relatively recent field of study that incorporates aspects of symplectic, complex, and Poisson geometry. As mentioned in Chapter 2, every symplectic or complex structure on a manifold induces a GC structure. Furthermore, every GC structure induces a Poisson structure, and if the GC structure was derived from a symplectic one, then this Poisson structure is the same one induced by the symplectic structure.

These connections to classical geometric structures naturally suggest for several avenues of potential study in GC geometry. One such is the pursuit of a useful definition of Hamiltonian group actions on GC manifolds, inspired by equivariant symplectic and Poisson geometry, as well as the underlying physics. Several independent groups of researchers have contributed to this search, including Bursztyn, Cavalcanti, and Gualtieri, [BCG07]; Hu, [Hu09]; Lin and Tolman, [LT06]; Stiénon and Xu, [SX08]; and Vaisman, [Vai07]. Because of the common sources of inspiration, unsurprisingly there are similarities between many these different approaches. In this thesis, we follow the approach of [LT06].

Lin and Tolman's construction generalizes the usual symplectic definition, in the sense that a Hamiltonian group actions in the symplectic sense are alsoHamiltonian with respect to the induced GC structure on the manifold. The Lin–Tolman reduction of GC manifolds by Hamiltonian group actions also generalizes the Symplectic Reduction Theorem, in the sense that the two versions of reduction commute with the association of a GC structure to a symplectic one.

Just as in the symplectic case, in order to ensure that the generalized reduced space is a manifold, one must make an assumption regarding freeness of the group action. Analogous to the work of Lerman and Sjamaar in [SL91], the main result of this chapter, Theorem 4.4.6, is that even if the reduction of a GC manifold by a Hamiltonian group action is a singular topological space, it decomposes into a union of GC manifolds. For the case of a *twisted* generalized complex manifold, we require the added constraint that the original manifold be compact.

A similar, although distinct, situation was studied in [JRS09], in which the authors considered the singular reduction of Dirac manifolds. They analyzed the global quotient of a Dirac manifold by a proper group action as a differential space, as in [CŚ01], and obtained conditions that guarantee the Dirac structure will descend to the quotient space.

An interesting difference between the symplectic and GC situations is that the generic result of symplectic reduction is a space with at worst orbifold singularities, whereas for the reduction of a *twisted* GC manifold, the generic result seems to be a space with worse-than-orbifold singularities. See Remark 4.4.1 below.

Section 4.1 is a rapid introduction to some essential notions from generalized complex geometry. Section 4.2 reviews some important facts about equivariant cohomology and the orbit type stratification of G-spaces. Section 4.3 consists of a summary of Hamiltonian actions and reduction in generalized complex geometry. Finally, Section 4.4 contains the full statement and proof of the main theorem of this chapter.

4.1 Generalized complex geometry

We begin by giving several standard definitions and results from generalized complex geometry, which can be found in [Gua03] or [BBB04].

4.1.1 Generalized complex linear algebra

For any real finite-dimensional vector space V, the space $V \oplus V^*$ carries a natural non-degenerate, symmetric bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ of signature (n, n), defined by

$$\langle\!\langle u + \alpha, v + \beta \rangle\!\rangle := \frac{1}{2} \left(\alpha(v) + \beta(u) \right)$$

for all $u, v \in V$, $\alpha, \beta \in V^*$. We will use the same notation for the complex bilinear extension of this bilinear form to the complexification $(V \oplus V^*) \otimes_{\mathbb{R}} \mathbb{C}$. (Note that this yields a complex bilinear form, rather than a sesquilinear form.) These bilinear forms will henceforth be referred to as the **standard metrics** on $V \oplus V^*$ and $(V \oplus V^*) \otimes \mathbb{C}$.

Lemma 4.1.1. *Let V be a real vector space.*

(a) Let $T \colon V \to V$ be a linear map, and denote its complex linear extension by T also. If $\lambda \in \mathbb{C}$ is an eigenvalue of T with eigenspace $V_\lambda \subset V \otimes \mathbb{C}$, then $\overline{\lambda}$ is an eigenvalue of T with eigenspace $\overline{V_\lambda} \subset V \otimes \mathbb{C}$.

(b) Let $J \colon V \to V$ be a complex structure on V, i.e. $J^2 = -\mathrm{id}_V$. Then J is diagonalizable with eigenvalues $\pm i$.

Proof.

(a) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of T, and let $\overline{v} \in \overline{V_\lambda}$. Since T is the complex linear extension of a real linear map, we have

$$T(\overline{v}) = \overline{T(v)} = \overline{\lambda v} = \overline{\lambda}\,\overline{v},$$

so $\overline{\lambda}$ is also an eigenvalue of T and $\overline{V_\lambda} \subset V_{\overline{\lambda}}$. On the other hand, if $w \in V_{\overline{\lambda}}$, then

$$T(\overline{w}) = \overline{T(w)} = \overline{\overline{\lambda} w} = \lambda \overline{v},$$

so $V_{\overline{\lambda}} \subset \overline{V_\lambda}$. Thus $V_{\overline{\lambda}} = \overline{V_\lambda}$.

(b) First, suppose $\lambda \in \mathbb{C}$ is an eigenvalue of the complex linear extension of J, which we denote by the same symbol. Then there is some nonzero $v \in V \otimes \mathbb{C}$ such that

$$-v = (J)^2(v) = J(\lambda v) = \lambda^2 v,$$

so $\lambda^2 = -1$, so $\lambda = \pm i$.

Now, note that any $v \in V \otimes \mathbb{C}$ can be decomposed as

$$v = \frac{v - iJ(v)}{2} + \frac{v + iJ(v)}{2},$$

and

$$J\left(\frac{v \mp iJ(v)}{2}\right) = \frac{J(v) \pm iv}{2} = \pm i \cdot \left(\frac{v \mp iJ(v)}{2}\right).$$

Therefore the eigenspaces of J span $V \otimes \mathbb{C}$, which means J is diagonalizable.

□

Proposition 4.1.2. *Let V be a real vector space. There is a natural bijective correspondence between the following two structures.*

(a) Complex linear subspaces $E \subset (V \oplus V^*) \otimes \mathbb{C}$ such that $E \cap \overline{E} = \{0\}$ and E is maximally isotropic with respect to the standard metric on $(V \oplus V^*) \otimes \mathbb{C}$.

(b) *Linear automorphisms $\mathcal{J}$ of $V \oplus V^*$ such that $\mathcal{J}^2 = -\mathrm{id}_{V \oplus V^*}$ and $\mathcal{J}$ is orthogonal with respect to the standard metric on $V \oplus V^*$.*

Proof. **1. $\Rightarrow$ 2.** Let $E \subset (V \oplus V^*) \otimes \mathbb{C}$ be a maximally isotropic complex linear subspace such that $E \cap \overline{E} = \{0\}$. Since E is maximally isotropic, we have $\dim_{\mathbb{C}} E = \frac{1}{2} \dim_{\mathbb{C}}((V \oplus V^*) \otimes \mathbb{C})$. Since $\dim_{\mathbb{C}} E = \dim_{\mathbb{C}} \overline{E}$ and $E \cap \overline{E} = \{0\}$, it follows that $(V \oplus V^*) \otimes \mathbb{C} = E \oplus \overline{E}$. Furthermore, because the standard metric on $(V \oplus V^*) \otimes \mathbb{C}$ is the complex linear extension of the one on $V \oplus V^*$, an easy but tedious computation proves that $\overline{E}$ is isotropic as well.

Define the complex linear map $\mathcal{J}_{\mathbb{C}} \colon (V \oplus V^*) \otimes \mathbb{C} \to (V \oplus V^*) \otimes \mathbb{C}$ by setting $\mathcal{J}_{\mathbb{C}}$ equal to multiplication by $+i$ on E and multiplication by $-i$ on $\overline{E}$. Since E is a complex linear subspace, we know $i \cdot E = E$ and $-i \cdot \overline{E} = \overline{i \cdot E} = \overline{E}$. Note also that for $x \in E$, $y \in \overline{E}$,

$$\begin{aligned}(\mathcal{J}_{\mathbb{C}})^2(x+y) &= \mathcal{J}_{\mathbb{C}}(ix - iy) \\ &= i^2 x + i^2 y \\ &= -(x+y),\end{aligned}$$

so $(\mathcal{J}_{\mathbb{C}})^2 = -\mathrm{id}$.

Since $(V \oplus V^*) \otimes \mathbb{C} = E \oplus \overline{E}$ and $V \oplus V^* = \{\, x + \overline{x} \mid x \in (V \oplus V^*) \otimes \mathbb{C} \,\}$, we can write

$$V \oplus V^* = \{\, e + \overline{e} \mid e \in E \,\}.$$

For all $e \in E$, we have $\mathcal{J}_{\mathbb{C}}(e + \overline{e}) = ie - i\overline{e} = ie + \overline{ie} \in V \oplus V^*$. Therefore $\mathcal{J}_{\mathbb{C}}$ preserves $V \oplus V^*$. Denote the restriction of $\mathcal{J}_{\mathbb{C}}$ to $V \oplus V^*$ by $\mathcal{J}$.

An arbitrary element of $(V \oplus V^*) \otimes \mathbb{C}$ can be written as $(x + \overline{x}) + i(y + \overline{y})$

for $x, y \in E$, and

$$\begin{aligned}\mathcal{J}_{\mathbb{C}}\left(x+\overline{x}+iy+i\overline{y}\right) &= ix - i\overline{x} + i^2 y - i^2\overline{y} \\ &= \mathcal{J}(x+\overline{x}) + i\,\mathcal{J}(y+\overline{y}),\end{aligned}$$

so $\mathcal{J}_{\mathbb{C}}$ is determined by its restriction $\mathcal{J}$ to $V \oplus V^*$. The property of squaring to the negative identity is preserved by restriction, so $\mathcal{J}$ is a complex structure on $V \oplus V^*$. Finally, using the fact that both E and $\overline{E}$ are isotropic, we compute

$$\begin{aligned}\langle\!\langle \mathcal{J}(x+\overline{x}), \mathcal{J}(y+\overline{y})\rangle\!\rangle &= \langle\!\langle ix - i\overline{x}, iy - i\overline{y}\rangle\!\rangle \\ &= \langle\!\langle ix, iy\rangle\!\rangle + \langle\!\langle -i\overline{x}, iy\rangle\!\rangle + \langle\!\langle ix, -i\overline{y}\rangle\!\rangle + \langle\!\langle -i\overline{x}, -i\overline{y}\rangle\!\rangle \\ &= 0 + -i^2\ \langle\!\langle \overline{x}, y\rangle\!\rangle + -i^2\ \langle\!\langle x, \overline{y}\rangle\!\rangle + 0 \\ &= \langle\!\langle x, y\rangle\!\rangle + \langle\!\langle \overline{x}, y\rangle\!\rangle + \langle\!\langle x, \overline{y}\rangle\!\rangle + \langle\!\langle \overline{x}, \overline{y}\rangle\!\rangle \\ &= \langle\!\langle x+\overline{x}, y+\overline{y}\rangle\!\rangle\end{aligned}$$

for all $x, y \in E$, so $\mathcal{J}$ is orthogonal.

2. $\Rightarrow$ **1.** Let $\mathcal{J}$ be an orthogonal complex structure on $V \oplus V^*$, and let $\mathcal{J}_{\mathbb{C}}$ denote its complex linear extension to $(V \oplus V^*) \otimes \mathbb{C}$. Because $(\mathcal{J}_{\mathbb{C}})^2 = -\mathrm{id}$, we know its eigenvalues are $\pm i$. Let E be the $(+i)$-eigenspace of $(V \oplus V^*) \otimes \mathbb{C}$ with respect to $\mathcal{J}_{\mathbb{C}}$. Since $\mathcal{J}_{\mathbb{C}}$ is orthogonal,

$$\langle\!\langle e, f\rangle\!\rangle = \langle\!\langle \mathcal{J}_{\mathbb{C}}(e), \mathcal{J}_{\mathbb{C}}(f)\rangle\!\rangle = \langle\!\langle ie, if\rangle\!\rangle = i^2\ \langle\!\langle e, f\rangle\!\rangle = -\ \langle\!\langle e, f\rangle\!\rangle$$

and hence $\langle\!\langle e, f\rangle\!\rangle = 0$ for all $e, f \in E$. Therefore E is isotropic.

By Lemma 4.1.1, we know that $\overline{E}$ is the $(-i)$-eigenspace of $(V \oplus V^*) \otimes \mathbb{C}$ with respect to $\mathcal{J}_{\mathbb{C}}$, and that $(V \oplus V^*) \otimes \mathbb{C} = E \oplus \overline{E}$. Therefore $E \cap \overline{E} = \{0\}$, and since E and $\overline{E}$ have the same complex dimension, their common dimension must be $\frac{1}{2}\dim_C((V \oplus V^*) \otimes \mathbb{C})$. Therefore E (and $\overline{E}$) are maximal isotropic.

$\square$

Definition 4.1.3. Let V be a real vector space. Either of the equivalent structures described in Proposition 4.1.2 will be called a **linear GC structure** on V. If V is equipped with a linear GC structure, then V is a **GC vector space**.

Let $E \subset (V \oplus V^*) \otimes \mathbb{C}$ be a linear GC structure on V, and let

$$\pi\colon (V \oplus V^*) \otimes \mathbb{C} \cong (V \otimes \mathbb{C}) \oplus (V^* \otimes \mathbb{C}) \to V \otimes \mathbb{C}$$

be the projection. The **type** of this GC structure is the complex codimension of $\pi(E)$ in $V \otimes \mathbb{C}$:

$$\operatorname{type}(E) = \dim_{\mathbb{C}}(V \otimes \mathbb{C}) - \dim_{\mathbb{C}} \pi(E).$$

Remark 4.1.4. Let V be a real vector space. A maximally isotropic subspace of $V \oplus V^*$, respectively $(V \oplus V^*) \otimes \mathbb{C}$, is called a **linear Dirac structure**, respectively **complex linear Dirac structure**, on V. Thus, a linear GC structure on V is a complex linear Dirac structure E on V such that $E \cap \overline{E} = \{0\}$.

Proposition 4.1.5. *Let $(V, \mathcal{J})$ be a GC vector space. Then $\dim V$ is even.*

Proof. Let

$$(V \oplus V^*)_+ := \{\, x \in V \oplus V^* \mid \langle\!\langle x, x \rangle\!\rangle > 0 \,\} \cup \{0\}$$

and

$$(V \oplus V^*)_- := \{\, x \in V \oplus V^* \mid \langle\!\langle x, x \rangle\!\rangle < 0 \,\} \cup \{0\},$$

and note that these are both vector subspaces of $V \oplus V^*$, and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is positive definite on $(V \oplus V^*)_+$ and negative definite on $(V \oplus V^*)_-$. Since the natural metric on $V \oplus V^*$ is symmetric, nondegenerate, and has signature (n, n), we know that these vector spaces each have dimension n, and their direct sum is $V \oplus V^*$. Furthermore, because $\mathcal{J}$ is orthogonal, both of these vector spaces are stable under

$\mathcal{J}$. The restriction of $\mathcal{J}$ to $(V \oplus V^*)_\pm$ gives each the structure of a complex vector space, so their dimensions must be even. Therefore n is even. $\square$

Example 4.1.6.

(a) Let (V, Ω) be a **symplectic vector space**, meaning that $\Omega \in \bigwedge^2(V^*)$ is a non-degenerate, skew-symmetric bilinear form on V. Define the map $\mathcal{J}_\Omega \colon V \oplus V^* \to V \oplus V^*$ by

$$\mathcal{J}_\Omega := \begin{pmatrix} 0 & -\Omega^\sharp \\ \Omega^\flat & 0 \end{pmatrix}.$$

As described in Section 4.1 of [Gua03], this is a linear GC structure on V of type 0, with linear Dirac structure

$$E_\Omega = \left\{ X - i\,\Omega^\flat(X) \;\middle|\; X \in V \otimes \mathbb{C} \right\}.$$

(b) Let (V, I) be a **complex vector space**, meaning that $I \colon V \to V$ is a linear map such that $I^2 = -\mathrm{id}_V$. Define the map $\mathcal{J}_I \colon V \oplus V^* \to V \oplus V^*$ by

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}.$$

As described in Section 4.1 of [Gua03], this is a linear GC structure on V of type $\frac{1}{2}\dim V$, with linear Dirac structure

$$E_I = V_{0,1} \oplus V^*_{1,0},$$

where $V_{1,0}, V_{0,1} \subset V \otimes \mathbb{C}$ denote the $(+i)$ and $(-i)$-eigenspaces of I, respectively.

Example 4.1.7. Let $(V_1, \mathcal{J}_1)$ and $(V_2, \mathcal{J}_2)$ be real GC vector spaces. Then the direct sum of $\mathcal{J}_1$ and $\mathcal{J}_2$ is a map

$$(\mathcal{J}_1, \mathcal{J}_2) \colon V^1 \oplus V_1^* \oplus V_2 \oplus V_2^* \to V_1 \oplus V_1^* \oplus V_2 \oplus V_2^*,$$

which under the identification

$$V_1 \oplus V_1^* \oplus V_2 \oplus V_2^* \cong (V_1 \oplus V_2) \oplus (V_1 \oplus V_2)^*$$

yields a linear GC structure on $V_1 \oplus V_2$. We will call this the **direct sum** of the linear GC structures on V_1 and V_2.

Suppose $G\colon V \oplus V^* \to V \oplus V^*$ is an orthogonal and involutive linear map, so that $G^2 = \mathrm{id}_V$. Then the associated bilinear form

$$(\mathcal{X}, \mathcal{Y}) \mapsto \langle\!\langle G(\mathcal{X}), \mathcal{Y} \rangle\!\rangle$$

on $V \oplus V^*$ is symmetric. We will call G **positive definite** if its associated bilinear form is positive definite, i.e. if $\langle\!\langle G(\mathcal{X}), \mathcal{X} \rangle\!\rangle > 0$ for all nonzero $\mathcal{X} \in V \oplus V^*$. Note that if $\mathcal{J}_1$ and $\mathcal{J}_2$ are commuting linear GC structures, then $G := -\mathcal{J}_1 \circ \mathcal{J}_2 \colon V \oplus V^* \to V \oplus V^*$ is orthogonal and involutive.

Definition 4.1.8. Let V be a real vector space. A **linear GK structure** on V is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting linear GC structures on V such that $G := -\mathcal{J}_1 \circ \mathcal{J}_2$ is positive definite. In this case $(V, \mathcal{J}_1, \mathcal{J}_2)$ is called a **GK vector space**.

Example 4.1.9. Let V be a **Kähler vector space**, so that V has both a linear symplectic structure Ω and a complex structure I, and the bilinear form on V defined by $(u, v) \mapsto g(u, v) := \Omega\left(u, I(v)\right)$ is symmetric and positive definite. Then $\Omega^\flat \circ I = -I^* \circ \Omega^\flat$ and $g^\flat = I^* \circ \Omega^\flat$. It follows that $\mathcal{J}_\Omega$ and $\mathcal{J}_I$ commute and

$$G := -\mathcal{J}_\Omega \circ \mathcal{J}_I = \begin{pmatrix} 0 & g^\sharp \\ g^\flat & 0 \end{pmatrix},$$

which is positive definite. Therefore $(\mathcal{J}_\Omega, \mathcal{J}_I)$ is a linear GK structure on V.

Definition 4.1.10. Let V be a real vector space, and let $B \in \bigwedge^2(V^*)$. The **B-transform** of $V \oplus V^*$ defined by B is the map

$$e^B\colon V \oplus V^* \to V \oplus V^*, \quad e^B := \begin{pmatrix} 1 & 0 \\ B^\flat & 1 \end{pmatrix}.$$

Proposition 4.1.11 (Section 2.2 of [Gua03]). *Let V be a real vector space and let $B \in \bigwedge^2(V^*)$. The B-field transform e^B is orthogonal with respect to the standard metrics on $V \oplus V^*$ and $(V \oplus V^*) \otimes \mathbb{C}$. It transforms linear GC structures on V by*

$$\mathcal{J} \mapsto e^B \circ \mathcal{J} \circ e^{-B} \quad \textit{and} \quad E \mapsto e^B(E)$$

for a linear GC structure given equivalently by a map $\mathcal{J}$ or a Dirac structure E, and it preserves types. It transforms linear GK structures $(\mathcal{J}_1, \mathcal{J}_2)$ by transforming $\mathcal{J}_1$ and $\mathcal{J}_2$ individually.

Proposition 4.1.12 ([BBB04], Lemma 8.2). *Let V be a real vector space with linear GC structure $E \subset (V \oplus V^*) \otimes \mathbb{C}$, and let $W \subset V$ be a linear subspace. Then the subspace $E_W \subset (W \oplus W^*) \otimes \mathbb{C}$ defined by*

$$E_W := \{\, u + \alpha|_W \in (W \oplus W^*) \otimes \mathbb{C} \mid u + \alpha \in (W_{\mathbb{C}} \oplus V \otimes \mathbb{C}^*) \cap E \,\}$$

is a complex linear Dirac structure on W.

The following definition comes from [BBB04].

Definition 4.1.13. Let (V, E) be a real GC vector space, and let $W \subset V$ be a vector subspace. If the subspace $E_W \subset (W \oplus W^*) \otimes \mathbb{C}$ satisfies $E_W \cap \overline{E_W} = \{0\}$, so that E_W is a GC structure for W, then we call (W, E_W) a **GC subspace** of (V, E), and denote by $\mathcal{J}_W$ the complex structure on $W \oplus W^*$ induced by E_W.

Definition 4.1.14. Let V be a real vector space with linear GC structure given by $\mathcal{J}\colon V \oplus V^* \to V \oplus V^*$, and let $W \subset V$ be a linear subspace. We say W is **split**

if there is a linear subspace $N \subset V$ such that $V = W \oplus N$, and $W \oplus \mathrm{Ann}(N)$ is preserved by $\mathcal{J}$. In this case, the subspace N is called a **splitting** for W in V.

Proposition 4.1.15 (Proposition 4.8 of [BBB04])**.** *Let V be a real vector space with linear GC structure given by $\mathcal{J}\colon V \oplus V^* \to V \oplus V^*$, let $W \subset V$ be a split linear subspace of V with splitting $N \subset V$. Then the following hold.*

(a) Both W and N are GC subspaces of V, and $V = W \oplus N$ is a direct sum of linear GC structures.

(b) Let $\psi\colon W \oplus \mathrm{Ann}(N) \to W \oplus W^*$ be the natural isomorphism given by $(w, f) \mapsto (w, f|_W)$. Then

$$\mathcal{J}_W = \psi \circ \left(\mathcal{J}|_{W \oplus \mathrm{Ann}(N)}\right) \circ \psi^{-1},$$

where $\mathcal{J}_W$ is the linear GC structure of W as a GC subspace of $(V, \mathcal{J})$.

(c) The subspace W is a splitting for N, and the above results hold with the roles of W and N reversed.

Corollary 4.1.16. *Let $(V, \mathcal{J}_1, \mathcal{J}_2)$ be a GK vector space, and let $W \subset V$ be a linear subspace. If $N \subset V$ is a splitting for N with respect to both $\mathcal{J}_1$ and $\mathcal{J}_2$, then $(W, (\mathcal{J}_1)_W, (\mathcal{J}_2)_W)$ is a GK vector space.*

Proof. Because $\mathcal{J}_1|_{W \oplus \mathrm{Ann}(N)}$ and $\mathcal{J}_2|_{W \oplus \mathrm{Ann}(N)}$ commute, it follows from part (b) of Proposition 4.1.15 that $(\mathcal{J}_1)_W$ and $(\mathcal{J}_2)_W$ commute, so $-(\mathcal{J}_1)_W \circ (\mathcal{J}_2)_W$ is orthogonal and involutive. That it is positive definite also follows from part (b) of the above proposition, and the fact that the isomorphism $\Psi\colon W \oplus \mathrm{Ann}(N) \to W \oplus W^*$ is isometric with respect to the restriction of the natural metric on $V \oplus V^*$ to the domain of ψ and the natural metric on $W \oplus W^*$. $\square$

Definition 4.1.17. Let $(V, \mathcal{J})$ be a GC vector space, and let G be a group acting linearly on V. Recall that the action of G on V induces a dual action of G on V^*, defined by $g \cdot f := f \circ g^{-1}$ for all $g \in G$ and $f \in V^*$. Therefore G acts on $V \oplus V^*$ by $g \mapsto (g, (g^{-1})^*)$ for all $g \in G$. We say the G-action on $(V, \mathcal{J})$ is **canonical** if the action of G on $V \oplus V^*$ commutes with $\mathcal{J}$, i.e. if the diagram

$$\begin{tikzcd} V \oplus V^* \arrow[r, "\mathcal{J}"] \arrow[d, "{(g,(g^{-1})^*)}"'] & V \oplus V^* \arrow[d, "{(g,(g^{-1})^*)}"] \\ V \oplus V^* \arrow[r, "\mathcal{J}"'] & V \oplus V^* \end{tikzcd}$$

commutes for all $g \in G$.

Lemma 4.1.18. *Let V be a real vector space with linear GC structure given by $E \subset (V \oplus V^*) \otimes \mathbb{C}$. Let G be a group acting linearly on V. Then G acts canonically on (V, E) if and only if E is stable under the complex linear extension of the G-action to $(V \oplus V^*) \otimes \mathbb{C}$.*

Proof. Let $\mathcal{J}$ be the complex structure on $V \oplus V^*$ induced by E, and suppose that the action of G on $V \oplus V^*$ commutes with J. Then the complex linear extension of $\mathcal{J}$ and the G-action to $(V \oplus V^*) \otimes \mathbb{C}$ commute also. Recall from the proof of Proposition 4.1.2 that E is the $(+i)$-eigenspace of $\mathcal{J}$. Let $v \in E$. Then for any $g \in G$, we have

$$\mathcal{J}(g \cdot v) = g \cdot \mathcal{J}(v) = g \cdot (iv) = i\,(g \cdot v),$$

so $g \cdot v \in E$. Therefore the action of G preserves E.

Now, assume the action of G on $(V \oplus V^*) \otimes \mathbb{C}$ preserves E, and let $\mathcal{J}$ be the complex structure on $V \oplus V^*$ induced by E. Because E is maximally isotropic with respect to the standard type (n, n) metric on $(V \oplus V^*) \otimes \mathbb{C}$, it has complex dimension n. Since $E \cap \overline{E} = \{0\}$, it follows that $(V \oplus V^*) \otimes \mathbb{C} = E \oplus \overline{E}$. The complexification $\mathcal{J}_{\mathbb{C}}$ of $\mathcal{J}$ is defined by making E the $(+i)$-eigenspace of $\mathcal{J}_{\mathbb{C}}$, and $\overline{E}$ the $(-i)$-eigenspace. Note that since the G-action on $(V \oplus V^*) \otimes \mathbb{C}$ preserves

E, it must also preserve $\overline{E}$. Let $v \in V \oplus V^*$, and write $v = x + y$ for some $x \in E$ and $y \in \overline{E}$. Then for any $g \in G$, we have

$$\begin{aligned} g \cdot \mathcal{J}(v) &= g \cdot \mathcal{J}_{\mathbb{C}}(v) \\ &= g \cdot (ix - iy) \\ &= i\,(g \cdot x) - i\,(g \cdot y) \\ &= \mathcal{J}_{\mathbb{C}}(g \cdot x) + \mathcal{J}_{\mathbb{C}}(g \cdot y) \\ &= \mathcal{J}_{\mathbb{C}}\left((g \cdot (x + y)\right) \\ &= \mathcal{J}(g \cdot v). \end{aligned}$$

Therefore the action of G commutes with $\mathcal{J}$. $\square$

Example 4.1.19. Let (V, Ω) be a symplectic vector space, let $\Omega^\flat \colon V \to V^*$ be the induced linear isomorphism, and let

$$\mathcal{J}_\Omega := \begin{pmatrix} 0 & -\Omega^\sharp \\ \Omega^\flat & 0 \end{pmatrix}$$

be the induced linear GC structure. Let G be a group acting linearly on V. This action is called **symplectic** with respect to Ω if $\Omega(g \cdot u, g \cdot v) = \Omega(u, v)$ for all $g \in G$ and $u, v \in V$. It is easy to check that the G-action is symplectic if and only if the map $\Omega^\flat \colon V \to V^*$ is G-equivariant with respect to the action on V and the induced action on V^*, if and only if the G-action commutes with $\mathcal{J}_\Omega$.

Lemma 4.1.20. *Let G be a compact Lie group acting linearly on a finite-dimensional real vector space V. Recall the induced action of G on the dual V^* of V, defined by*

$$g \cdot \lambda \colon v \mapsto \lambda(g^{-1} \cdot v)$$

for all $g \in G$, $\lambda \in V^*$, and $v \in V$. Then there is a G-equivariant linear isomorphism $V \to V^*$, and thus the dimensions of the G-fixed point sets in V and V^* are *the same:*

$$\dim V^G = \dim(V^*)^G.$$

Proof. Choose a G-invariant inner product on V, whose constructibility is assured by the compactness of G. Then the action of G on V is orthogonal with respect to this inner product. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V with respect to this inner product, and let $\{\lambda_1, \ldots, \lambda_n\}$ denote the corresponding dual basis, which satisfies $\langle \lambda_i, v_j \rangle = \delta_{ij}$ for all $i, j = 1, \ldots n$, where $\langle \cdot, \cdot \rangle$ denotes the pairing $V^* \times V \to \mathbb{R}$ and δ_{ij} is the Kronecker delta. Define the linear isomorphism $T \colon V \to V^*$ by $T(v_i) = \lambda_i$ for each $i = 1, \ldots, n$.

With respect to the ordered basis for V given above, if the transformation corresponding to an element $g \in G$ has matrix $[g_{ij}]_{i,j=1}^n$, then because this transformation is orthogonal we know that the matrix $((g^{-1})_{ij})_{i,j=1}^n$ for the transformation corresponding to g^{-1} satisfies $(g^{-1})_{ij} = g_{ji}$ for all $i, j = 1, \ldots, n$. For any $i, j = 1, \ldots, n$ and $g \in G$ we have

$$\langle T(g \cdot v_i), v_j \rangle = \left\langle T\left(\sum_{k=1}^n g_{ik} v_k\right), v_j \right\rangle = \sum_{k=1}^n g_{ik} \langle \lambda_k, v_j \rangle = g_{ij},$$

and

$$\begin{aligned}
\langle g \cdot T(v_i), v_j \rangle &= \left\langle \lambda_i, g^{-1} \cdot v_j \right\rangle \\
&= \left\langle \lambda_i, \sum_{k=1}^n (g^{-1})_{jk} v_k \right\rangle \\
&= \left\langle \lambda_i, \sum_{k=1}^n g_{kj} v_k \right\rangle \\
&= \sum_{k=1}^n g_{kj} \langle \lambda_i, v_k \rangle \\
&= g_{ij}.
\end{aligned}$$

Thus T is G-equivariant. □

Proposition 4.1.21. *Let $(V, \mathcal{J})$ be a GC vector space, let G be a compact topological group that acts canonically on $(V, \mathcal{J})$, and let*

$$V^G := \{\, v \in V \mid g \cdot v = v \text{ for all } g \in G \,\}$$

be the set of G-fixed points of V. Then V^G is split.

Proof. Let dg be a bi-invariant Haar measure on G, adjusted so that $dg(G) = 1$. Define the function $\pi \colon V \to V^G$ by

$$\pi(v) := \int_G (g \cdot v)\, dg$$

for $v \in V$. It is easy to check that π is well-defined, linear, and acts as the identity on V^G. Set $N := \ker \pi$. We claim that N is a splitting for V^G in V.

The map π is a linear projection from V onto V^G, so $V = V^G \oplus \ker \pi = V^G \oplus N$. Observe also that N is G-stable, because for all $v \in N$ and $h \in G$,

$$\pi(h \cdot v) = \int_G g \cdot (h \cdot v)\, dg = \int_G (gh) \cdot v\, dg = \int_G (g \cdot v)\, dg = \pi(v) = 0,$$

since dg is right-invariant.

To complete the proof of this proposition, it remains to show that $V^G \oplus \mathrm{Ann}(N)$ is preserved by $\mathcal{J}$. We claim that $\mathrm{Ann}(N) = (V^*)^G$, which would imply that $(V \oplus V^*)^G = V^G \oplus (V^*)^G = V^G \oplus \mathrm{Ann}(N)$. Since $\mathcal{J}$ commutes with the action of G on $V \oplus V^*$, we know that $\mathcal{J}\left((V \oplus V^*)^G\right) = (V \oplus V^*)^G$, and hence a proof of this claim completes the proof of this proposition.

Let $\lambda \in \mathrm{Ann}(N)$, let $g \in G$, and let $u \in V$. Decompose u as $u = v + w$ for

$v \in V^G$ and $w \in N$. Then

$$\begin{aligned}
(g \cdot \lambda)(u) &= \lambda(g^{-1} \cdot u) \\
&= \lambda(g^{-1} \cdot v) + \lambda(g^{-1} \cdot w) \\
&= \lambda(v) + \lambda(g^{-1} \cdot w) && \text{since } v \in V^G \\
&= \lambda(v) + 0 && \text{since } w \in N,\ N \text{ is } G\text{-stable, and } \lambda \in \operatorname{Ann}(N) \\
&= \lambda(v) + \lambda(w) && \text{since } w \in N \text{ and } \lambda \in \operatorname{Ann}(N) \\
&= \lambda(u).
\end{aligned}$$

Therefore $\lambda \in (V^*)^G$, so $\operatorname{Ann}(N) \subset (V^*)^G$. Note that

$$\operatorname{Ann}(N) \cong (V/N)^* = \big((V^G \oplus N)/N\big)^* \cong (V^G)^*.$$

Hence $\dim \operatorname{Ann}(N) = \dim(V^G) = \dim(V^*)^G$, so $\operatorname{Ann}(N) = (V^*)^G$. □

4.1.2 Generalized complex manifolds

For any smooth manifold M, the **generalized tangent bundle**, also known as the **Pontryagin bundle** or **big tangent bundle**, of M is $\mathbb{T}M := \mathsf{T}M \oplus \mathsf{T}^*M$. This vector bundle carries a natural, smoothly-varying, fiber-wise, non-degenerate, symmetric bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ of signature (n, n), defined by

$$\langle\!\langle u + \alpha, v + \beta \rangle\!\rangle := \frac{1}{2}\left(\alpha(v) + \beta(u)\right)$$

for all $x \in M$ and $u + \alpha, v + \beta \in \mathbb{T}_x M$. We will use the same notation for the complex bilinear extension of this metric to the complexification $\mathbb{T}_{\mathbb{C}}M := \mathbb{T}M \otimes_{\mathbb{R}} \mathbb{C}$. These metrics will henceforth be referred to as the **standard metrics** on $\mathbb{T}M$ and $\mathbb{T}_{\mathbb{C}}M$.

The following result follows from Proposition 4.1.2.

Proposition 4.1.22. *Let M be a manifold. There is a natural bijective correspondence between the following two structures.*

(a) Complex linear subbundles $E \subset \mathbb{T}_{\mathbb{C}}M$ over M such that $E \cap \overline{E} = 0$ and E is maximally isotropic with respect to the standard metric on $\mathbb{T}_{\mathbb{C}}V$. (Here 0 denotes the image of the zero section of $\mathbb{T}_{\mathbb{C}}M \to M$, as is customary.)

(b) Bundle automorphisms $\mathcal{J}$ of $\mathbb{T}M$ over id_M such that $\mathcal{J}^2 = -\mathrm{id}$ and $\mathcal{J}$ is orthogonal with respect to the standard metric on $\mathbb{T}M$.

Definition 4.1.23. Let M be a manifold. Either of the equivalent structures described in Proposition 4.1.22 will be called an **almost GC structure** on M. If M is equipped with an almost GC structure, then M is an **almost GC manifold**.

Let $E \subset \mathbb{T}_{\mathbb{C}}M$ be an almost GC structure on M, and for each $x \in M$ let $\pi_x \colon \mathbb{T}_{\mathbb{C},x}M \to \mathsf{T}_{\mathbb{C},x}M$ be the projection. The **type** of this almost GC structure at the point $x \in M$ is the complex codimension of $\pi_x(E_x)$ in $\mathsf{T}_{\mathbb{C},x}M$:

$$\mathrm{type}(E)_x = \dim_{\mathbb{C}} \mathsf{T}_{\mathbb{C},x}M - \dim_{\mathbb{C}} \pi_x(E_x).$$

An **almost GK structure** on M is a pair of commuting almost GC structures $(\mathcal{J}_1, \mathcal{J}_2)$ on M such that the orthogonal and involutive bundle map $G := -\mathcal{J}_1 \circ \mathcal{J}_2$ is **positive definite**, in the sense that $G|_{\mathbb{T}_xM} \colon \mathbb{T}_xM \to \mathbb{T}_xM$ is positive definite for each $x \in M$.

Remark 4.1.24. Let M be a manifold. A maximally isotropic linear subbundle of $\mathbb{T}M$, respectively $\mathbb{T}_{\mathbb{C}}M$, is called a **Dirac structure**, respectively **complex Dirac structure** on M. Thus an almost GC structure on M is a complex Dirac structure $E \subset \mathbb{T}_{\mathbb{C}}M$ satisfying $E \cap \overline{E} = 0$, (where here 0 refers to the image of the zero section of $\mathbb{T}_{\mathbb{C}}M \to M$, as is usual).

The Lie bracket defines a skew-symmetric bilinear bracket on sections of the tangent bundle $\mathsf{T}M$. This can be extended to a skew-symmetric bilinear bracket on sections of the generalized tangent bundle $\mathbb{T}M$, called the **Courant bracket**, defined by

$$[X+\alpha, Y+\beta] := [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}\,\mathrm{d}\,(\beta(X) - \alpha(Y))$$

for all $X+\alpha, Y+\beta \in \Gamma(\mathbb{T}M)$, where $\Gamma(\mathbb{T}M)$ denotes the space of smooth sections of $\mathbb{T}M \to M$. Here the bracket on the right-hand side is the usual Lie bracket of vector fields, and $\mathcal{L}$ denotes Lie differentiation. For each closed differential three-form $H \in \Omega^3_{\mathrm{cl}}(M)$, there is also the ***H*-twisted Courant bracket**, defined by

$$[X+\alpha, Y+\beta]_H := [X+\alpha, Y+\beta] + \iota_Y\iota_X H$$

$X+\alpha, Y+\beta \in \Gamma(\mathbb{T}M)$. Both the Courant and the H-twisted Courant brackets extend complex linearly to brackets on smooth sections of the complexified generalized tangent bundle $\mathbb{T}_{\mathbb{C}}M$, which will be denoted the same way.

Definition 4.1.25. Let M be a manifold, and let L be a real (respectively complex) linear subbundle of $\mathbb{T}M$ (respectively $\mathbb{T}_{\mathbb{C}}M$). Then L is **Courant involutive** if the space $\Gamma(L)$ of smooth sections of L is closed under the Courant bracket, i.e. $\big[\Gamma(L), \Gamma(L)\big] \subset \Gamma(L)$. If $H \in \Omega^3_{\mathrm{cl}}(M)$, we similarly define ***H*-twisted Courant involutive**.

Let $E \subset \mathbb{T}_{\mathbb{C}}M$ be an almost GC structure on M. This is a **GC structure** if E is Courant involutive, in which case (M, E) is a **GC manifold**. If $H \in \Omega^3_{\mathrm{cl}}(M)$ and E is H-twisted Courant involutive, then E is an ***H*-twisted GC structure**, and (M, E, H) is a **twisted GC manifold**.

Let $(\mathcal{J}_1, \mathcal{J}_2)$ be an almost GK structure on M. This is a **GK structure** if both $\mathcal{J}_1$ and $\mathcal{J}_2$ are Courant involutive, in which case $(M, \mathcal{J}_1, \mathcal{J}_2)$ is a **GK manifold**.

If $H \in \Omega^3_{\mathrm{cl}}(M)$ and $\mathcal{J}_1$ and $\mathcal{J}_2$ are H-twisted Courant involutive, then this is an **H-twisted GK structure**, and $(M, \mathcal{J}_1, \mathcal{J}_2, H)$ is a **twisted GK manifold**.

Remark 4.1.26. Let M be a manifold and D be a real of complex Dirac structure on M. Then D is called **closed**, or **integrable**, if the space $\Gamma(D)$ of smooth sections of D is Courant involutive. Thus a GC structure on M is a closed complex Dirac structure $E \subset \mathbb{T}_{\mathbb{C}}M$ such that $E \cap \overline{E} = 0$.

The integrability condition for almost GC structures $\mathcal{J}$ on a manifold M can be stated in terms of the **Courant–Nijenhuis torsor** of $\mathcal{J}$, which is the $\mathbb{R}$-bilinear map $N_{\mathcal{J}} \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ defined by

$$N_{\mathcal{J}}(\mathcal{X}, \mathcal{Y}) := [\mathcal{X}, \mathcal{Y}] + \mathcal{J}\left([\mathcal{J}(\mathcal{X}), \mathcal{Y}] + [\mathcal{X}, \mathcal{J}(\mathcal{Y})]\right) - [\mathcal{J}(\mathcal{X}), \mathcal{J}(\mathcal{Y})]$$

for $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathbb{T}M)$. For $H \in \Omega^2_{\mathrm{cl}}(M)$, one can analogously define the **H-twisted Courant–Nijenhuis torsor** by using the H-twisted Courant bracket instead of the untwisted one.

Proposition 4.1.27. *Let $(M, \mathcal{J})$ be an almost GC manifold, and let $H \in \Omega^3_{cl}(M)$. Then the $(+i)$-eigenbundle $E \subset \mathbb{T}_{\mathbb{C}}M$ of $\mathcal{J}$ is H-twisted Courant involutive if and only if the H-twisted Courant–Nijenhuis torsor $N_{\mathcal{J}}$ vanishes identically on M.*

Proof. Let $N_{\mathcal{J}}$ denote both the H-twisted Courant–Nijenhuis torsor and its complex linear extension to a $\mathbb{C}$-bilinear map $\Gamma(\mathbb{T}_{\mathbb{C}}M) \times \Gamma(\mathbb{T}_{\mathbb{C}}M) \to \Gamma(\mathbb{T}_{\mathbb{C}}M)$. Observe that if $\mathcal{Z}, \mathcal{W} \in \Gamma(E)$, then

$$\begin{aligned} N_{\mathcal{J}}(\mathcal{Z}, \mathcal{W}) &= [\mathcal{Z}, \mathcal{W}]_H + \mathcal{J}\left([\mathcal{J}(\mathcal{Z}), \mathcal{W}]_H + [\mathcal{Z}, \mathcal{J}(\mathcal{W})]_H\right) - [\mathcal{J}(\mathcal{Z}), \mathcal{J}(\mathcal{W})]_H \\ &= [\mathcal{Z}, \mathcal{W}]_H + \mathcal{J}\left([i\,\mathcal{Z}, \mathcal{W}]_H + [\mathcal{Z}, i\,\mathcal{W}]_H\right) - [i\,\mathcal{Z}, i\,\mathcal{W}]_H \\ &= [\mathcal{Z}, \mathcal{W}]_H + \mathcal{J}\left(2i\,[\mathcal{Z}, \mathcal{W}]_H\right) - i^2\,[\mathcal{Z}, \mathcal{W}]_H \\ &= 2\,[\mathcal{Z}, \mathcal{W}]_H + 2i\,\mathcal{J}\left([\mathcal{Z}, \mathcal{W}]_H\right). \end{aligned} \tag{4.1}$$

Therefore, if $N_{\mathcal{J}}$ vanishes on $\gamma(\mathbb{T}M)$, then its extension to $\gamma(\mathbb{T}_{\mathbb{C}}M)$ vanishes also, so

$$2\ [\mathcal{Z},\mathcal{W}]_H + 2i\,\mathcal{J}\left([\mathcal{Z},\mathcal{W}]_H\right) = 0$$

and hence

$$\mathcal{J}\left([\mathcal{Z},\mathcal{W}]_H\right) = i\ [\mathcal{Z},\mathcal{W}]_H$$

for all $\mathcal{Z},\mathcal{W}\in\Gamma(E)\subset\Gamma(\mathbb{T}_{\mathbb{C}}M)$. This means that $[\Gamma(E),\Gamma(E)]\subset\Gamma(E)$.

Suppose now that E is H-twisted Courant involutive. Then by (4.1) we have

$$\begin{aligned} N_{\mathcal{J}}(\mathcal{Z},\mathcal{W}) &= 2\ [\mathcal{Z},\mathcal{W}]_H + 2i\,\mathcal{J}\left([\mathcal{Z},\mathcal{W}]_H\right) \\ &= 2\ [\mathcal{Z},\mathcal{W}]_H + 2i^2\ [\mathcal{Z},\mathcal{W}]_H \\ &= 0 \end{aligned} \tag{4.2}$$

for any $\mathcal{Z},\mathcal{W}\in\Gamma(E)$. It is easy to check that the complex-linear extensions to $\Gamma(\mathbb{T}_{\mathbb{C}}M)$ of the GC structure $\mathcal{J}$, the H-twisted Courant bracket, and hence also the Courant–Nijenhuis torsor $N_{\mathcal{J}}$ on $\Gamma(\mathbb{T}M)$ are compatible with conjugation, in the sense that

$$\begin{aligned} \mathcal{J}(\overline{\mathcal{Z}}) &= \overline{\mathcal{J}(\mathcal{Z})}, \\ [\overline{\mathcal{Z}},\overline{\mathcal{W}}]_H &= \overline{[\mathcal{Z},\mathcal{W}]_H},\ \text{and} \\ N_{\mathcal{J}}(\overline{\mathcal{Z}},\overline{\mathcal{W}}) &= \overline{N_{\mathcal{J}}(\mathcal{Z},\mathcal{W})} \end{aligned}$$

for all $\mathcal{Z},\mathcal{W}\in\Gamma(\mathbb{T}_{\mathbb{C}}M)$. If $\mathcal{Z},\mathcal{W}\in\Gamma(\overline{E})$, then $\overline{\mathcal{Z}},\overline{\mathcal{W}}\in\Gamma(E)$, so by (4.2) we have

$$\overline{N_{\mathcal{J}}(\mathcal{Z},\mathcal{W})} = N_{\mathcal{J}}(\overline{\mathcal{Z}},\overline{\mathcal{W}}) = 0,$$

and hence $N_{\mathcal{J}}(\mathcal{Z},\mathcal{W}) = 0$ as well. Finally, if $\mathcal{Z}\in\Gamma(E)$ and $\mathcal{W}\in\Gamma(\overline{E})$, then since

$\overline{E}$ is the $(-i)$-eigenbundle of $\mathcal{J}$, we have

$$\begin{aligned}
N_{\mathcal{J}}(\mathcal{Z},\mathcal{W}) &= [\mathcal{Z},\mathcal{W}]_H + \mathcal{J}\left([\mathcal{J}(\mathcal{Z}),\mathcal{W}]_H + [\mathcal{Z},\mathcal{J}(\mathcal{W})]_H\right) - [\mathcal{J}(\mathcal{Z}),\mathcal{J}(\mathcal{W})]_H \\
&= [\mathcal{Z},\mathcal{W}]_H + \mathcal{J}\left([i\,\mathcal{Z},\mathcal{W}]_H + [\mathcal{Z},-i\,\mathcal{W}]_H\right) - [i\,\mathcal{Z},-i\,\mathcal{W}]_H \\
&= [\mathcal{Z},\mathcal{W}]_H + \mathcal{J}\left(i\;[\mathcal{Z},\mathcal{W}]_H - i\;[\mathcal{Z},\mathcal{W}]_H\right) + i^2\;[\mathcal{Z},\mathcal{W}]_H \\
&= 0,
\end{aligned}$$

and similarly $N_{\mathcal{J}}(\mathcal{W},\mathcal{Z}) = 0$. Therefore $N_{\mathcal{J}}$ vanishes when restricted to $\Gamma(E) \times \Gamma(E)$, $\Gamma(\overline{E}) \times \Gamma(\overline{E})$, $\Gamma(E) \times \Gamma(\overline{E})$, and $\Gamma(\overline{E}) \times \Gamma(E)$. Since $\mathbb{T}_{\mathbb{C}}M = E \oplus \overline{E}$, this means that $N_{\mathcal{J}}$ vanishes on $\Gamma(\mathbb{T}_{\mathbb{C}}M)$, and thus also on $\Gamma(\mathbb{T}M) \subset \Gamma(\mathbb{T}_{\mathbb{C}}M)$. □

Remark 4.1.28. Observe that the Courant–Nijenhuis torsor is identical in form to the *Nijenhuis tensor* for almost complex structures, except that it uses the Courant bracket instead of the Lie bracket, and is applied to sections of the generalized tangent bundle rather than the tangent bundle. By the Newlander-Nirenberg Theorem, an almost complex structure corresponds to a complex structure on the manifold exactly when the Nijenhuis tensor vanishes identically. By the same reasoning as in the proof of Proposition 4.1.27, this is equivalent to the $(+i)$-eigenbundle with respect to the almost complex structure being closed under the complexification of the Lie bracket. Thus the integrability condition for almost GC structures is completely analogous to the integrability condition for almost complex structures.

Definition 4.1.29. Let M be a manifold, and let $B \in \Omega^2(M)$, where $\Omega^2(M)$ denotes the space of differential two-forms on M. The B**-transform** of $\mathbb{T}M$ defined by B is the map

$$e^B : \mathbb{T}M \to \mathbb{T}M, \quad e^B := \begin{pmatrix} 1 & 0 \\ B^\flat & 1 \end{pmatrix}.$$

The B-transform e^B is called **closed** or **exact** if the two-form B is closed or exact, respectively.

The proposition below follows almost immediately from Proposition 4.1.11 above.

Proposition 4.1.30. *Let M be a manifold and let $B \in \Omega^2(M)$. The B-field transform e^B is orthogonal with respect to the standard metrics on $\mathbb{T}M$ and $\mathbb{T}_{\mathbb{C}}M$. It transforms almost GC structures on V by*

$$\mathcal{J} \mapsto e^B \circ \mathcal{J} \circ e^{-B} \quad \text{and} \quad E \mapsto e^B(E)$$

for a almost GC structure given equivalently by a map $\mathcal{J}$ or a Dirac structure E, and it preserves types. It transforms almost GK structures $(\mathcal{J}_1, \mathcal{J}_2)$ by transforming $\mathcal{J}_1$ and $\mathcal{J}_2$ individually.

Proposition 4.1.31 (Proposition 3.23 of [Gua03])**.** *Let M be a manifold, let $H \in \Omega^3_{cl}(M)$, and let $B \in \Omega^2(M)$. The B-transform of an H-twisted GC structure is an $(H + \mathrm{d}B)$-twisted GC structure. Thus, a closed B-transform of an untwisted GC structure is untwisted.*

Remark 4.1.32. For a manifold M, the symmetries of $\mathbb{T}M$ are the bundle automorphisms $F\colon \mathbb{T}M \to \mathbb{T}M$ covering diffeomorphisms $\phi\colon M \to M$ such that F preserves both the standard metric on $\mathbb{T}M$ and the Courant bracket on $\Gamma(\mathbb{T}M)$.

Given a diffeomorphism $\phi\colon M \to M$, we can obtain a symmetry of $\mathbb{T}M$ by setting $F = (\phi_*, (\phi^{-1})^*)$. However, there are symmetries of $\mathbb{T}M$, and hence of generalized geometric structures, which do not arise in this way. These extra symmetries are precisely the closed B-transforms.

Compare this to the case of symmetries of $\mathsf{T}M$, which are bundle automorphisms $F\colon \mathsf{T}M \to \mathsf{T}M$ covering diffeomorphisms $\phi\colon M \to M$ such that F preserves just the Lie bracket on $\Gamma(\mathsf{T}M)$. In this case, it must be that $F = \phi_*$.

Proofs of these assertions can be found in §3.3 of [Gua03].

Example 4.1.33.

(a) Let (M, ω) be a pre-symplectic manifold, meaning that $\omega \in \Omega^2(M)$ is a non-degenerate form on M, but *not necessarily closed*. Just as in the linear case, this defines an almost GC structure $\mathcal{J}_\omega$ on M by

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^\sharp \\ \omega^\flat & 0 \end{pmatrix}$$

of type 0 at every point. It has associated Dirac structure defined by

$$E_{\omega,x} = \left\{ X - i\,\omega_x^\flat(X) \;\middle|\; X \in \mathsf{T}_{\mathbb{C},x}M \right\},$$

for each $x \in M$. Its Courant–Nijenhuis torsor $\mathcal{N}_{\mathcal{J}_\omega}$ is given in Proposition 4.1.34 below, from which it follows that it is a GC structure if and only if $\mathrm{d}\omega = 0$, i.e. if and only if ω is a symplectic structure on M.

(b) Let (M, I) be an almost complex manifold, meaning that $I^2 = -\mathrm{id}_{\mathsf{T}M}$ but I is *not necessarily integrable*. Just as in the linear case, this defines an almost GC structure $\mathcal{J}_I$ on M by

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}.$$

of type $\frac{1}{2}\dim M$ at every point. It has associated Dirac structure defined by

$$E_I = \mathsf{T}_{0,1}M \oplus \mathsf{T}^*_{1,0}M,$$

where $\mathsf{T}_{1,0}M, \mathsf{T}_{0,1}M \subset \mathsf{T}_{\mathbb{C}}M$ denote the $(+i)$ and $(-i)$-eigenbundles of I, respectively. Its Courant–Nijenhuis torsor $\mathcal{N}_{\mathcal{J}_I}$ is given in Proposition 4.1.35 below, from which it follows that it is a GC structure if and only if I is integrable, i.e. if and only if I is a complex structure on M.

(c) Let M be a **Kähler manifold** with Kähler form $\omega \in \Omega^2(M)$, complex structure $I\colon \mathsf{T}M \to \mathsf{T}M$, and associated Riemannian metric g. Then just as in the linear case $\mathcal{J}_\omega$ and $\mathcal{J}_I$ commute and

$$G := -\mathcal{J}_\omega \circ \mathcal{J}_I = \begin{pmatrix} 0 & g^\sharp \\ g^\flat & 0 \end{pmatrix}$$

is positive definite, so $(M, \mathcal{J}_\omega, \mathcal{J}_I)$ is a GK manifold.

In the following two proofs, we will make frequent use of the following formula for operators on differential forms. For vector fields X and Y, we have

$$\iota_{[X,Y]} = \iota_X \mathcal{L}_Y - \mathcal{L}_Y \iota_X \text{ (and hence } \iota_X \mathcal{L}_Y = \mathcal{L}_Y \iota_X + \iota_{[X,Y]}).$$

Proposition 4.1.34. *Let M be a manifold and let $\omega \in \Omega^2(M)$ be a pre-symplectic structure. Then the Courant–Nijenhuis torsor of the associated almost GC structure $\mathcal{J}_\omega$ is given by*

$$\mathcal{N}_{\mathcal{J}_\omega}(X+\alpha, Y+\beta) = \left(\omega^\sharp \iota_{\omega^\sharp \beta} \iota_{\omega^\sharp \alpha} - \omega^\sharp \iota_Y \iota_X + \iota_{\omega^\sharp \beta} \iota_X - \iota_{\omega^\sharp \alpha} \iota_Y\right) \mathrm{d}\omega$$

for $X+\alpha, Y+\beta \in \Gamma(\mathbb{T}M)$.

Proof. Throughout this proof we will use the fact that, for $\lambda \in \Omega^1(M)$ and $v \in \mathrm{Vec}(M)$, we have

$$\langle \lambda, v \rangle = \left\langle \omega^\flat \omega^\sharp \lambda, v \right\rangle = \omega(\omega^\sharp \lambda, v).$$

Write $\mathcal{J} = \mathcal{J}_\omega$, and let $\mathcal{X} = X+\alpha$, $\mathcal{Y} = Y+\beta \in \Gamma(\mathbb{T}M)$. Then $\mathcal{J}(\mathcal{X}) = \mathcal{J}(X+\alpha) = -\omega^\sharp \alpha + \omega^\flat X$ and $\mathcal{J}(\mathcal{Y}) = \mathcal{J}(Y+\beta) = -\omega^\sharp \beta + \omega^\flat Y$. We compute

$$
\begin{aligned}
[\mathcal{X},\mathcal{Y}] &= [X,Y]+\mathcal{L}_X\beta-\mathcal{L}_Y\alpha-\frac{1}{2}\mathrm{d}\left(\langle\beta,X\rangle-\langle\alpha,Y\rangle\right),\\
[\mathcal{J}\mathcal{X},\mathcal{Y}] &= -[\omega^\sharp\alpha,Y]-\mathcal{L}_{\omega^\sharp\alpha}\beta-\mathcal{L}_Y(\omega^\flat X)-\frac{1}{2}\mathrm{d}\left(-\left\langle\beta,\omega^\sharp\alpha\right\rangle-\left\langle\omega^\flat X,Y\right\rangle\right),\\
[\mathcal{X},\mathcal{J}\mathcal{Y}] &= -[X,\omega^\sharp\beta]+\mathcal{L}_X(\omega^\flat Y)+\mathcal{L}_{\omega^\sharp\beta}\alpha-\frac{1}{2}\mathrm{d}\left(\left\langle\omega^\flat Y,X\right\rangle+\left\langle\alpha,\omega^\sharp\beta\right\rangle\right),\ \text{and}\\
[\mathcal{J}\mathcal{X},\mathcal{J}\mathcal{Y}] &= [\omega^\sharp\alpha,\omega^\sharp\beta]-\mathcal{L}_{\omega^\sharp\alpha}(\omega^\flat Y)+\mathcal{L}_{\omega^\sharp\beta}(\omega^\flat X)\\
&\qquad-\frac{1}{2}\mathrm{d}\left(-\left\langle\omega^\flat Y,\omega^\sharp\alpha\right\rangle+\left\langle\omega^\flat X,\omega^\sharp\beta\right\rangle\right).
\end{aligned}
$$

Since

$$
\left\langle\omega^\flat Y,\omega^\sharp\alpha\right\rangle=\omega(Y,\omega^\sharp\alpha)=-\omega(\omega^\sharp\alpha,Y)=-\langle\alpha,Y\rangle,
$$

and similarly $\left\langle\omega^\flat X,\omega^\sharp\beta\right\rangle=-\langle\beta,X\rangle$, we know

$$
\begin{aligned}
&[\mathcal{X},\mathcal{Y}]-[\mathcal{J}\mathcal{X},\mathcal{J}\mathcal{Y}]\\
&\qquad=[X,Y]-[\omega^\sharp\alpha,\omega^\sharp\beta]+\mathcal{L}_X\beta-\mathcal{L}_Y\alpha+\mathcal{L}_{\omega^\sharp\alpha}(\omega^\flat Y)-\mathcal{L}_{\omega^\sharp\beta}(\omega^\flat X)\\
&\qquad\quad-\frac{1}{2}\mathrm{d}\left(\langle\beta,X\rangle-\langle\alpha,Y\rangle\right)+\frac{1}{2}\mathrm{d}\left(\langle\alpha,Y\rangle-\langle\beta,X\rangle\right)\\
&\qquad=[X,Y]-[\omega^\sharp\alpha,\omega^\sharp\beta]+\mathcal{L}_X\beta-\mathcal{L}_Y\alpha+\mathcal{L}_{\omega^\sharp\alpha}(\omega^\flat Y)-\mathcal{L}_{\omega^\sharp\beta}(\omega^\flat X)\\
&\qquad\quad-\mathrm{d}\left(\langle\beta,X\rangle-\langle\alpha,Y\rangle\right).
\end{aligned}
$$

Also

$$
\begin{aligned}
&[\mathcal{J}\mathcal{X},\mathcal{Y}]+[\mathcal{X},\mathcal{J}\mathcal{Y}]\\
&\qquad=-[\omega^\sharp\alpha,Y]-[X,\omega^\sharp\beta]-\mathcal{L}_{\omega^\sharp\alpha}\beta-\mathcal{L}_Y(\omega^\flat X)+\mathcal{L}_X(\omega^\flat Y)+\mathcal{L}_{\omega^\sharp\beta}\alpha\\
&\qquad\quad-\frac{1}{2}\mathrm{d}\left(-\omega(\omega^\sharp\beta,\omega^\sharp\alpha)-\omega(X,Y)+\omega(Y,X)+\omega(\omega^\sharp\alpha,\omega^\sharp\beta)\right)\\
&\qquad=-[\omega^\sharp\alpha,Y]-[X,\omega^\sharp\beta]-\mathcal{L}_{\omega^\sharp\alpha}\beta-\mathcal{L}_Y(\omega^\flat X)+\mathcal{L}_X(\omega^\flat Y)+\mathcal{L}_{\omega^\sharp\beta}\alpha\\
&\qquad\quad+\mathrm{d}\left(\omega(X,Y)-\omega(\omega^\sharp\alpha,\omega^\sharp\beta)\right).
\end{aligned}
$$

We will compute the vector and covector parts of $\mathcal{N}_{\mathcal{J}}(\mathcal{X},\mathcal{Y})$ separately.

The vector part of

$$\mathcal{N}_{\mathcal{J}}(\mathcal{X},\mathcal{Y}) = [\mathcal{X},\mathcal{Y}] + \mathcal{J}\left([\mathcal{J}(\mathcal{X}),\mathcal{Y}] + [\mathcal{X},\mathcal{J}(\mathcal{Y})]\right) - [\mathcal{J}(\mathcal{X}),\mathcal{J}(\mathcal{Y})]\,,$$

is

$$\mathbf{Vec}\left([\mathcal{X},\mathcal{Y}] - [\mathcal{J}\mathcal{X},\mathcal{J}\mathcal{Y}]\right) - \omega^{\sharp}\left(\mathbf{Covec}\left([\mathcal{J}\mathcal{X},\mathcal{Y}] + [\mathcal{X},\mathcal{J}\mathcal{Y}]\right)\right).$$

We begin by simplifying $\mathbf{Covec}\left([\mathcal{J}\mathcal{X},\mathcal{Y}] + [\mathcal{X},\mathcal{J}\mathcal{Y}]\right)$.

Let Z be a vector field. Then

$$\begin{aligned}
&\iota_Z \mathbf{Covec}\left([\mathcal{J}\mathcal{X},\mathcal{Y}] + [\mathcal{X},\mathcal{J}\mathcal{Y}]\right)\\
&\qquad = -\iota_Z\mathcal{L}_{\omega^\sharp\alpha}\beta - \iota_Z\mathcal{L}_Y(\omega^\flat X) + \iota_Z\mathcal{L}_X(\omega^\flat Y) + \iota_Z\mathcal{L}_{\omega^\sharp\beta}\alpha\\
&\qquad\quad + \iota_Z \mathrm{d}\left(\omega(X,Y) - \omega(\omega^\sharp\alpha,\omega^\sharp\beta)\right)\\
&\qquad = -\left(\mathcal{L}_{\omega^\sharp\alpha}\iota_Z + \iota_{[Z,\omega^\sharp\alpha]}\right)\beta - \left(\mathcal{L}_Y\iota_Z + \iota_{[Z,Y]}\right)(\omega^\flat X)\\
&\qquad\quad + \left(\mathcal{L}_X\iota_Z + \iota_{[Z,X]}\right)(\omega^\flat Y) + \left(\mathcal{L}_{\omega^\sharp\beta}\iota_Z + \iota_{[Z,\omega^\sharp\beta]}\right)\alpha\\
&\qquad\quad + \mathcal{L}_Z\left(\omega(X,Y)\right) - \mathcal{L}_Z\left(\omega(\omega^\sharp\alpha,\omega^\sharp\beta)\right)\\
&\qquad = -\mathcal{L}_{\omega^\sharp\alpha}\langle\beta,Z\rangle - \left\langle\beta,[Z,\omega^\sharp\alpha]\right\rangle - \mathcal{L}_Y\left\langle\omega^\flat X,Z\right\rangle - \left\langle\omega^\flat X,[Z,Y]\right\rangle\\
&\qquad\quad + \mathcal{L}_X\left\langle\omega^\flat Y,Z\right\rangle + \left\langle\omega^\flat Y,[Z,X]\right\rangle + \mathcal{L}_{\omega^\sharp\beta}\langle\alpha,Z\rangle + \left\langle\alpha,[Z,\omega^\sharp\beta]\right\rangle\\
&\qquad\quad + \mathcal{L}_Z\left(\omega(X,Y)\right) - \mathcal{L}_Z\left(\omega(\omega^\sharp\alpha,\omega^\sharp\beta)\right)\\
&\qquad = -\mathcal{L}_{\omega^\sharp\alpha}\left(\omega(\omega^\sharp\beta,Z)\right) - \omega(\omega^\sharp\beta,[Z,\omega^\sharp\alpha])\\
&\qquad\quad - \mathcal{L}_Y\left(\omega(X,Z)\right) - \omega(X,[Z,Y])\\
&\qquad\quad + \mathcal{L}_X\left(\omega(Y,Z)\right) + \omega(Y,[Z,X])\\
&\qquad\quad + \mathcal{L}_{\omega^\sharp\beta}\left(\omega(\omega^\sharp\alpha,Z)\right) + \omega(\omega^\sharp\alpha,[Z,\omega^\sharp\beta])\\
&\qquad\quad + \mathcal{L}_Z\left(\omega(X,Y)\right) - \mathcal{L}_Z\left(\omega(\omega^\sharp\alpha,\omega^\sharp\beta)\right).
\end{aligned}$$

Using the notation $\displaystyle\sum_{\mathrm{cyclic}(X,Y,Z)}$ to denote the sum over all cyclic permutations of

(X, Y, Z), and similarly for $(\omega^\sharp\alpha, \omega^\sharp\beta, Z)$, we can write

$$
\begin{aligned}
&\iota_Z \mathbf{Covec}\left([\mathcal{J}\mathcal{X}, \mathcal{Y}] + [\mathcal{X}, \mathcal{J}\mathcal{Y}]\right) \\
&\qquad = \sum_{\text{cyclic}(X,Y,Z)} \left(\mathcal{L}_X\left(\omega(Y, Z)\right) + \omega(X, [Y, Z])\right) \\
&\qquad\qquad - \sum_{\text{cyclic}(\omega^\sharp\alpha, \omega^\sharp\beta, Z)} \left(\mathcal{L}_{\omega^\sharp\alpha}\left(\omega(\omega^\sharp\beta, Z)\right) + \omega(\omega^\sharp\alpha, [\omega^\sharp\beta, Z])\right) \\
&\qquad\qquad - \omega(Z, [X, Y]) + \omega(Z, [\omega^\sharp\alpha, \omega^\sharp\beta]) \\
&\qquad = (\mathrm{d}\omega)(X, Y, Z) - (\mathrm{d}\omega)(\omega^\sharp\alpha, \omega^\sharp\beta, Z) \\
&\qquad\qquad + \omega([X, Y], Z) - \omega([\omega^\sharp\alpha, \omega^\sharp\beta], Z) \\
&\qquad = \iota_Z\left(\iota_Y\iota_X(\mathrm{d}\omega) - \iota_{\omega^\sharp\beta}\iota_{\omega^\sharp\alpha}(\mathrm{d}\omega) + \omega^\flat([X, Y]) - \omega^\flat([\omega^\sharp\alpha, \omega^\sharp\beta])\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\omega^\sharp \mathbf{Covec}\left([\mathcal{J}\mathcal{X}, \mathcal{Y}] + [\mathcal{X}, \mathcal{J}\mathcal{Y}]\right) \\
&\qquad = \omega^\sharp\left(\iota_Y\iota_X(\mathrm{d}\omega) - \iota_{\omega^\sharp\beta}\iota_{\omega^\sharp\alpha}(\mathrm{d}\omega) + \omega^\flat([X, Y]) - \omega^\flat([\omega^\sharp\alpha, \omega^\sharp\beta])\right) \\
&\qquad = \omega^\sharp\left(\iota_Y\iota_X - \iota_{\omega^\sharp\beta}\iota_{\omega^\sharp\alpha}\right)(\mathrm{d}\omega) + [X, Y] - [\omega^\sharp\alpha, \omega^\sharp\beta],
\end{aligned}
$$

so the vector part of $\mathcal{N}_{\mathcal{J}}(\mathcal{X}, \mathcal{Y})$ is

$$
\begin{aligned}
&\mathbf{Vec}\left([\mathcal{X}, \mathcal{Y}] - [\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}]\right) - \omega^\sharp\left(\mathbf{Covec}\left([\mathcal{J}\mathcal{X}, \mathcal{Y}] + [\mathcal{X}, \mathcal{J}\mathcal{Y}]\right)\right) \\
&\quad = [X, Y] - [\omega^\sharp\alpha, \omega^\sharp\beta] - \omega^\sharp\left(\iota_Y\iota_X - \iota_{\omega^\sharp\beta}\iota_{\omega^\sharp\alpha}\right)(\mathrm{d}\omega) - [X, Y] + [\omega^\sharp\alpha, \omega^\sharp\beta] \\
&\quad = \omega^\sharp\left(\iota_{\omega^\sharp\beta}\iota_{\omega^\sharp\alpha} - \iota_Y\iota_X\right)(\mathrm{d}\omega).
\end{aligned} \tag{4.3}
$$

The covector part of $\mathcal{N}_{\mathcal{J}}(\mathcal{X}, \mathcal{Y})$ is

$$
\begin{aligned}
&\mathbf{Covec}\left([\mathcal{X}, \mathcal{Y}] - [\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}]\right) + \omega^\flat\left(\mathbf{Vec}\left([\mathcal{J}\mathcal{X}, \mathcal{Y}] + [\mathcal{X}, \mathcal{J}\mathcal{Y}]\right)\right) \\
&\qquad = \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \mathcal{L}_{\omega^\sharp\alpha}(\omega^\flat Y) - \mathcal{L}_{\omega^\sharp\beta}(\omega^\flat X) \\
&\qquad\qquad - \mathrm{d}\left(\langle\beta, X\rangle - \langle\alpha, Y\rangle\right) - \omega^\beta\left([\omega^\sharp\alpha, Y] + [X, \omega^\sharp\beta]\right).
\end{aligned}
$$

Let Z be a vector field. Then

$$
\begin{aligned}
\iota_Z \mathbf{Covec}\left([\mathcal{X},\mathcal{Y}] - [\mathcal{J}\mathcal{X},\mathcal{J}\mathcal{Y}]\right) &+ \iota_Z\omega^\flat\left(\mathbf{Vec}\left([\mathcal{J}\mathcal{X},\mathcal{Y}] + [\mathcal{X},\mathcal{J}\mathcal{Y}]\right)\right) \\
&= \iota_Z\mathcal{L}_X\beta - \iota_Z\mathcal{L}_Y\alpha + \iota_Z\mathcal{L}_{\omega^\sharp\alpha}(\omega^\flat Y) - \iota_Z\mathcal{L}_{\omega^\sharp\beta}(\omega^\flat X) \\
&\qquad - \iota_Z \mathrm{d}\left(\langle\beta, X\rangle - \langle\alpha, Y\rangle\right) - \iota_Z\omega^\beta\left([\omega^\sharp\alpha, Y] + [X, \omega^\sharp\beta]\right) \\
&= \left(\mathcal{L}_X\iota_Z + \iota_{[Z,X]}\right)\beta - \left(\mathcal{L}_Y\iota_Z + \iota_{[Z,Y]}\right)\alpha \\
&\qquad + \left(\mathcal{L}_{\omega^\sharp\alpha}\iota_Z + \iota_{[Z,\omega^\sharp\alpha]}\right)(\omega^\flat Y) - \left(\mathcal{L}_{\omega^\sharp\beta}\iota_Z + \iota_{[Z,\omega^\sharp\beta]}\right)(\omega^\flat X) \\
&\qquad - \mathcal{L}_Z\langle\beta, X\rangle + \mathcal{L}_Z\langle\alpha, Y\rangle - \omega([\omega^\sharp\alpha, Y], Z) - \omega([X, \omega^\sharp\beta], Z) \\
&= \mathcal{L}_X\langle\beta, Z\rangle + \langle\beta, [Z, X]\rangle - \mathcal{L}_Y\langle\alpha, Z\rangle - \langle\alpha, [Z, Y]\rangle \\
&\qquad + \mathcal{L}_{\omega^\sharp\alpha}\left\langle\omega^\flat Y, Z\right\rangle + \left\langle\omega^\flat Y, [Z, \omega^\sharp\alpha]\right\rangle \\
&\qquad - \mathcal{L}_{\omega^\sharp\beta}\left\langle\omega^\flat X, Z\right\rangle - \left\langle\omega^\flat X, [Z, \omega^\sharp\beta]\right\rangle \\
&\qquad - \mathcal{L}_Z\langle\beta, X\rangle + \mathcal{L}_Z\langle\alpha, Y\rangle - \omega([\omega^\sharp\alpha, Y], Z) - \omega([X, \omega^\sharp\beta], Z) \\
&= \mathcal{L}_X\left(\omega(\omega^\sharp\beta, Z)\right) + \omega(\omega^\sharp\beta, [Z, X]) \\
&\qquad - \mathcal{L}_Y\left(\omega(\omega^\sharp\alpha, Z)\right) - \omega(\omega^\sharp\alpha, [Z, Y]) \\
&\qquad + \mathcal{L}_{\omega^\sharp\alpha}\left(\omega(Y, Z)\right) + \omega(Y, [Z, \omega^\sharp\alpha]) \\
&\qquad - \mathcal{L}_{\omega^\sharp\beta}\left(\omega(X, Z)\right) - \omega(X, [Z, \omega^\sharp\beta]) \\
&\qquad - \mathcal{L}_Z\left(\omega(\omega^\sharp\beta, X)\right) + \mathcal{L}_Z\left(\omega(\omega^\sharp\alpha, Y)\right) \\
&\qquad - \omega([\omega^\sharp\alpha, Y], Z) - \omega([X, \omega^\sharp\beta], Z). \\
&= \sum_{\mathrm{cyclic}(X,\omega^\sharp\beta,Z)} \left(\mathcal{L}_X\left(\omega(\omega^\sharp\beta, Z)\right) + \omega(X, [\omega^\sharp\beta, Z])\right) \\
&\qquad - \sum_{\mathrm{cyclic}(Y,\omega^\sharp\alpha,Z)} \left(\mathcal{L}_Y\left(\omega(\omega^\sharp\alpha, Z)\right) + \omega(Y, [\omega^\sharp\alpha, Z])\right) \\
&= (\mathrm{d}\omega)(X, \omega^\sharp\beta, Z) - (\mathrm{d}\omega)(Y, \omega^\sharp\alpha, Z) \\
&= \left(\iota_Z\iota_{\omega^\sharp\beta}\iota_X - \iota_Z\iota_{\omega^\sharp\alpha}\iota_Y\right)(\mathrm{d}\omega) \\
&= \iota_Z\left(\iota_{\omega^\sharp\beta}\iota_X - \iota_{\omega^\sharp\alpha}\iota_Y\right)(\mathrm{d}\omega).
\end{aligned}
$$

Hence the covector part of $\mathcal{N}_{\mathcal{J}}(\mathcal{X}, \mathcal{Y})$ is

$$\begin{aligned}\mathbf{Covec}\left([\mathcal{X}, \mathcal{Y}] - [\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}]\right) + \omega^\flat\left(\mathbf{Vec}\left([\mathcal{J}\mathcal{X}, \mathcal{Y}] + [\mathcal{X}, \mathcal{J}\mathcal{Y}]\right)\right)& \\ = \left(\iota_{\omega^\sharp\beta}\iota_X - \iota_{\omega^\sharp\alpha}\iota_Y\right)(\mathrm{d}\omega).& \quad (4.4)\end{aligned}$$

Putting together (4.3) and (4.4), we have

$$\mathcal{N}_{\mathcal{J}}(\mathcal{X}, \mathcal{Y}) = \left(\omega^\sharp\iota_{\omega^\sharp\beta}\iota_{\omega^\sharp\alpha} - \omega^\sharp\iota_Y\iota_X + \iota_{\omega^\sharp\beta}\iota_X - \iota_{\omega^\sharp\alpha}\iota_Y\right)\mathrm{d}\omega,$$

as desired. □

Proposition 4.1.35. *Let M be a manifold and let $I\colon \mathsf{T}M \to \mathsf{T}M$ be an almost complex structure. Then the Courant–Nijenhuis torsor of the associated almost GC structure $\mathcal{J}_I$ is given by*

$$\mathcal{N}_{\mathcal{J}_I}(X + \alpha, Y + \beta) = N_I(X, Y) + \alpha\left(N_I(Y, \cdot)\right) - \beta\left(N_I(X, \cdot)\right)$$

for $X + \alpha$, $Y + \beta \in \Gamma(\mathbb{T}M)$.

Proof. Write $\mathcal{J} = \mathcal{J}_I$, and let $\mathcal{X} = X + \alpha$, $\mathcal{Y} = Y + \beta \in \Gamma(\mathbb{T}M)$. Then $\mathcal{J}(\mathcal{X}) = \mathcal{J}(X + \alpha) = -IX + I^*\alpha$ and $\mathcal{J}(\mathcal{Y}) = \mathcal{J}(Y + \beta) = -IY + I^*\beta$. We compute

$$\begin{aligned}
[\mathcal{X}, \mathcal{Y}] &= [X, Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}\mathrm{d}\left(\langle\beta, X\rangle - \langle\alpha, Y\rangle\right), \\
[\mathcal{J}\mathcal{X}, \mathcal{Y}] &= [-IX, Y] - \mathcal{L}_{IX}\beta - \mathcal{L}_Y(I^*\alpha) - \frac{1}{2}\mathrm{d}\left(-\langle\beta, IX\rangle - \langle I^*\alpha, Y\rangle\right) \\
&= -[IX, Y] - \mathcal{L}_{IX}\beta - \mathcal{L}_Y(I^*\alpha) + \frac{1}{2}\mathrm{d}\left(\langle\beta, IX\rangle + \langle\alpha, IY\rangle\right), \\
[\mathcal{X}, \mathcal{J}\mathcal{Y}] &= [X, -IY] + \mathcal{L}_X(I^*\beta) + \mathcal{L}_{IY}\alpha - \frac{1}{2}\mathrm{d}\left(\langle I^*\beta, X\rangle + \langle\alpha, IY\rangle\right) \\
&= -[X, IY] + \mathcal{L}_X(I^*\beta) + \mathcal{L}_{IY}\alpha - \frac{1}{2}\mathrm{d}\left(\langle\beta, IX\rangle + \langle\alpha, IY\rangle\right), \text{ and} \\
[\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}] &= [IX, IY] - \mathcal{L}_{IX}(I^*\beta) + \mathcal{L}_{IY}(I^*\alpha) - \frac{1}{2}\mathrm{d}\left(-\langle I^*\beta, IX\rangle + \langle I^*\alpha, IY\rangle\right) \\
&= [IX, IY] - \mathcal{L}_{IX}(I^*\beta) + \mathcal{L}_{IY}(I^*\alpha) - \frac{1}{2}\mathrm{d}\left(-\left\langle\beta, I^2X\right\rangle + \left\langle\alpha, I^2Y\right\rangle\right) \\
&= [IX, IY] - \mathcal{L}_{IX}(I^*\beta) + \mathcal{L}_{IY}(I^*\alpha) - \frac{1}{2}\mathrm{d}\left(\langle\beta, X\rangle - \langle\alpha, Y\rangle\right).
\end{aligned}$$

Therefore

$$[\mathcal{X},\mathcal{Y}]-[\mathcal{J}\mathcal{X},\mathcal{J}\mathcal{Y}]=[X,Y]-[IX,IY]+\mathcal{L}_X\beta+\mathcal{L}_{IX}(I^*\beta)-\mathcal{L}_Y\alpha-\mathcal{L}_{IY}(I^*\alpha)$$

and

$$\begin{aligned}
&\mathcal{J}\left([\mathcal{J}\mathcal{X},\mathcal{Y}]+[\mathcal{X},\mathcal{J}\mathcal{Y}]\right)\\
&\quad=-I\left(-[IX,Y]-[X,IY]\right)+I^*\left(-\mathcal{L}_{IX}\beta-\mathcal{L}_Y(I^*\alpha)+\mathcal{L}_X(I^*\beta)+\mathcal{L}_{IY}\alpha\right)\\
&\quad=I\left([IX,Y]+[X,IY]\right)+I^*\left(\mathcal{L}_X(I^*\beta)-\mathcal{L}_{IX}\beta+\mathcal{L}_{IY}\alpha-\mathcal{L}_Y(I^*\alpha)\right).
\end{aligned}$$

Then the vector part of

$$\mathcal{N}_{\mathcal{J}}(\mathcal{X},\mathcal{Y})=[\mathcal{X},\mathcal{Y}]+\mathcal{J}\left([\mathcal{J}\mathcal{X},\mathcal{Y}]+[\mathcal{X},\mathcal{J}\mathcal{Y}]\right)-[\mathcal{J}\mathcal{X},\mathcal{J}\mathcal{Y}]$$

is

$$[X,Y]-[IX,IY]+I\left([IX,Y]+[X,IY]\right)=N_I(X,Y), \tag{4.5}$$

and the covector part is

$$\begin{aligned}
&\mathcal{L}_X\beta+\mathcal{L}_{IX}(I^*\beta)-\mathcal{L}_Y\alpha-\mathcal{L}_{IY}(I^*\alpha)\\
&\qquad+I^*\left(\mathcal{L}_X(I^*\beta)\right)-I^*\left(\mathcal{L}_{IX}\beta\right)+I^*\left(\mathcal{L}_{IY}\alpha\right)-I^*\left(\mathcal{L}_Y(I^*\alpha)\right).
\end{aligned} \tag{4.6}$$

Applying the one-forms in (4.6) to an arbitrary vector field Z, we have

$$\begin{aligned}
\iota_Z\mathcal{L}_X\beta&=\mathcal{L}_X\left\langle\beta,Z\right\rangle+\left\langle\beta,[Z,X]\right\rangle,\\
\iota_Z\mathcal{L}_{IX}(I^*\beta)&=\mathcal{L}_{IX}\left\langle\beta,IZ\right\rangle+\left\langle\beta,I[Z,IX]\right\rangle,\\
\iota_Z\mathcal{L}_Y\alpha&=\mathcal{L}_Y\left\langle\alpha,Z\right\rangle+\left\langle\alpha,[Z,Y]\right\rangle,\\
\iota_Z\mathcal{L}_{IY}(I^*\alpha)&=\mathcal{L}_{IY}\left\langle\alpha,IZ\right\rangle+\left\langle\alpha,I[Z,IY]\right\rangle,
\end{aligned}$$

$$
\begin{aligned}
\iota_Z I^*\left(\mathcal{L}_X(I^*\beta)\right) &= \iota_{IZ}\mathcal{L}_X(I^*\beta)\\
&= \mathcal{L}_X\left\langle \beta, I^2 Z\right\rangle + \left\langle \beta, I[IZ,X]\right\rangle\\
&= -\mathcal{L}_X\left\langle \beta, Z\right\rangle + \left\langle \beta, I[IZ,X]\right\rangle,\\
\iota_Z I^*\left(\mathcal{L}_{IX}\beta\right) &= \iota_{IZ}\mathcal{L}_{IX}\beta\\
&= \mathcal{L}_{IX}\left\langle \beta, IZ\right\rangle + \left\langle \beta, [IZ,IX]\right\rangle,\\
\iota_Z I^*\left(\mathcal{L}_{IY}\alpha\right) &= \iota_{IZ}\mathcal{L}_{IY}\alpha\\
&= \mathcal{L}_{IY}\left\langle \alpha, IZ\right\rangle + \left\langle \alpha, [IZ,IY]\right\rangle, \text{ and}\\
\iota_Z I^*\left(\mathcal{L}_Y(I^*\alpha)\right) &= \iota_{IZ}\mathcal{L}_Y(I^*\alpha)\\
&= \mathcal{L}_Y\left\langle \alpha, I^2 Z\right\rangle + \left\langle \alpha, I[IZ,Y]\right\rangle\\
&= -\mathcal{L}_Y\left\langle \alpha, Z\right\rangle + \left\langle \alpha, I[IZ,Y]\right\rangle.
\end{aligned}
$$

Therefore ι_Z of the formula (4.6) equals

$$
\begin{aligned}
&\left\langle \beta, [Z,X] + I[Z,IX] + I[IZ,X] - [IZ,IX]\right\rangle\\
&\qquad - \left\langle \alpha, [Z,Y] + I[Z,IY] + I[IZ,Y] - [IZ,IY]\right\rangle\\
&= \left\langle \alpha, [Y,Z] + I[Y,IZ] + I[IY,Z] - [IY,IZ]\right\rangle\\
&\qquad - \left\langle \beta, [X,Z] + I[X,IZ] + I[IX,Z] - [IX,IZ]\right\rangle\\
&= \left\langle \alpha, N_I(Y,Z)\right\rangle - \left\langle \beta, N_I(X,Z)\right\rangle,
\end{aligned}
$$

so the covector part of $\mathcal{N}_{\mathcal{J}}(\mathcal{X},\mathcal{Y})$ is

$$
\alpha\left(N_I(Y,\cdot)\right) - \beta\left(N_I(X,\cdot)\right). \tag{4.7}
$$

Putting (4.5) and (4.7) together, we have

$$
\mathcal{N}_{\mathcal{J}}(\mathcal{X},\mathcal{Y}) = N_I(X,Y) + \alpha\left(N_I(Y,\cdot)\right) - \beta\left(N_I(X,\cdot)\right),
$$

as desired. □

Example 4.1.36. Let $(M_1, \mathcal{J}_1)$ and $(M_2, \mathcal{J}_2)$ be almost GC manifolds. Then the direct sum $\mathcal{J}$ of $\mathcal{J}_1$ and $\mathcal{J}_2$ is a map $\mathcal{J} := (\mathcal{J}_1, \mathcal{J}_2) \colon \mathbb{T}M_1 \oplus \mathbb{T}M_2 \to \mathbb{T}M_1 \oplus \mathbb{T}M_2$, which under the identification $\mathbb{T}M_1 \oplus \mathbb{T}M_2 \cong \mathbb{T}(M_1 \times M_2)$ yields an almost GC structure on $M_1 \times M_2$. We will call this the **direct sum** of the almost GC structures on M_1 and M_2. It is not hard to see that $(\mathcal{J}_1, \mathcal{J}_2)$ is a GC structure on $M_1 \times M_2$ if and only if $\mathcal{J}_i$ is a GC structure on M_i for $i = 1, 2$.

Let $H_1 \in \Omega^3_{\mathrm{cl}}(M_1)$ and $H_2 \in \Omega^3_{\mathrm{cl}}(M_2)$, let $\pi_i \colon M_1 \times M_2 \to M_i$ be the natural projection for $i = 1, 2$, and set $H := \pi_1^* H_1 + \pi_2^* H_2$. By the naturality of the exterior derivative, we know H is a closed three-form on $M_1 \times M_2$. Furthermore, it is not hard to see that $(\mathcal{J}_1, \mathcal{J}_2)$ is an H-twisted GC structure on $M_1 \times M_2$ if and only if $\mathcal{J}_i$ is an H_i-twisted GC structure on M_i for $i = 1, 2$.

There is a completely analogous product construction for almost GK and GK manifolds as well.

Let (M, E, H) be a twisted GC manifold. Suppose S is a submanifold of M given by the embedding $j \colon S \hookrightarrow M$. Although j induces a natural embedding $\mathsf{T}j \colon \mathsf{T}S \hookrightarrow \mathsf{T}M$ of tangent bundles, because of the contravariance of cotangent bundles there is in general no obvious embedding $\mathbb{T}S \hookrightarrow \mathbb{T}M$ of the generalized tangent bundles.

For each $x \in S$, define

$$E_{S,x} := \left\{ (X, \lambda|_S) \in \mathbb{T}_{\mathbb{C},x}S \;\middle|\; (X, \lambda) \in (\mathsf{T}_{\mathbb{C},x}S \oplus \mathsf{T}^*_{\mathbb{C},x}M) \cap E_x \right\}.$$

By Proposition 4.1.12, each $E_{S,x}$ is a complex linear Dirac structure on $\mathbb{T}_x S$. Let $E_S := \bigsqcup_{x \in S} E_{S,x}$. Then E_S is a constant-rank complex linear distribution of $\mathbb{T}_{\mathbb{C}}S$, but is not in general a smooth subbundle, nor will it generally satisfy $E_S \cap \overline{E_S} = 0$.

Proposition 4.1.37. *Let (M, E, H) be a twisted GC manifold, let $j\colon S \hookrightarrow M$ be a submanifold, and let $E_S \subset \mathbb{T}_{\mathbb{C}}S$ be as defined above. If E_S is a subbundle of $\mathbb{T}_{\mathbb{C}}S$, then E_S is (j^*H)-twisted Courant involutive.*

In the untwisted case, where $H = 0$, Proposition 4.1.37 was proved in [Cou90, Corollary 3.1.4]. The proof in the twisted case is nearly identical, with only minor changes to this proof and the relevant definitions and precursory results, (i.e. Definition 2.3.2, Propositions 2.3.3 and 3.1.3, and Corollary 3.1.4 in [Cou90]).

Definition 4.1.38. Let (M, E, H) be a twisted GC manifold, and let $j\colon S \hookrightarrow M$ be a submanifold. If $E_S \subset \mathbb{T}_{\mathbb{C}}S$ is a subbundle and satisfies $E_S \cap \overline{E_S} = 0$, then (S, E_S, j^*H) is a **(twisted) GC submanifold** of (M, E, H), and we denote by $\mathcal{J}_S$ the GC structure on S induced by E_S.

Remark 4.1.39. Suppose (M, E, H) is a twisted GC manifold, and $j\colon S \hookrightarrow M$ is an *open* submanifold. Then since $\mathsf{T}S$ and T^*S can be identified with $(\mathsf{T}M)|_S$ and $(\mathsf{T}^*M)|_S$, respectively, we see that we can identify E_S with $E|_S$, that $\mathcal{J}_S = \mathcal{J}|_{\mathbb{T}S}$, and that j^*H can be identified with $H|_S$. Therefore an open submanifold of an H-twisted GC manifold is automatically an H-twisted GC manifold. Similarly, an open submanifold of an H-twisted GK manifold is automatically an H-twisted GK manifold.

Definition 4.1.40. Let $(M, \mathcal{J})$ be an almost GC manifold, and let $S \subset M$ be a submanifold. A **splitting bundle** for S with respect to $(M, \mathcal{J})$ is a subbundle N of $\mathsf{T}M|_S \to S$ such that $\mathsf{T}M|_S = \mathsf{T}S \oplus N$ and $\mathsf{T}S \oplus \mathrm{Ann}(N) \subset \mathbb{T}M$ is invariant under $\mathcal{J}$. If a splitting bundle exists for S, then S is called a **split submanifold** of $(M, \mathcal{J})$.

The following is an extension of Proposition 5.12 of [BBB04] to the twisted case. The proofs of both the original and the twisted version essentially come down to Proposition 4.1.15.

Proposition 4.1.41. *Let $(M, \mathcal{J}, H)$ be a twisted GC manifold, and let $i\colon S \hookrightarrow M$ be a split submanifold of M with splitting bundle $N \to S$. Then S is an (i^*H)-twisted GC submanifold of $(M, \mathcal{J}, H)$, and the GC structure corresponding to the bundle E_S is the same as the one induced by the restriction of $\mathcal{J}$ via the natural isomorphism*

$$\mathbb{T}S \cong \mathsf{T}S \oplus \operatorname{Ann} N \subset \mathbb{T}M.$$

Corollary 4.1.42. *Let $(M, \mathcal{J}_1, \mathcal{J}_2, H)$ be a twisted GK manifold, and and let $i\colon S \hookrightarrow M$ be a split submanifold of M with respect to both $\mathcal{J}_1$ and $\mathcal{J}_2$, with splitting bundle $N \to S$. Then $(S, (\mathcal{J}_1)_W, (\mathcal{J}_2)_W)$ is an (i^*H)-twisted GK manifold.*

Definition 4.1.43. Let M be a manifold, and let G be a Lie group acting smoothly on M. This lifts to an action of G on $\mathbb{T}M$ by bundle automorphisms, given by

$$\left(g_*, (g^{-1})^*\right) : \mathsf{T}M \oplus \mathsf{T}^*M \to \mathsf{T}M \oplus \mathsf{T}^*M$$

for each $g \in G$, where g_* is the **pushforward** of tangent vectors by the map $g\colon M \to M$ and $(g^{-1})^*$ is the **pullback** of tangent covectors by the map $g^{-1}\colon M \to M$.

Let $\mathcal{J}$ be an H-twisted GC structure on M. We say that the G-action on $(M, \mathcal{J}, H)$ is **canonical** if the following hold.

(a) The differential form H is G-invariant, i.e. $g^*H = H$ for all $g \in G$.

(b) The action of G on $\mathbb{T}M$ commutes with $\mathcal{J}$, i.e. the diagram

$$\begin{tikzcd}
\mathbb{T}M \arrow[r, "\mathcal{J}"] \arrow[d, "{(g_*,(g^{-1})^*)}"'] & \mathbb{T}M \arrow[d, "{(g_*,(g^{-1})^*)}"] \\
\mathbb{T}M \arrow[r, "\mathcal{J}"'] & \mathbb{T}M
\end{tikzcd}$$

commutes for all $g \in G$.

It is easy to check that a smooth group action on a manifold commutes with an almost GC structure $\mathcal{J}\colon \mathbb{T}M \to \mathbb{T}M$ if and only if the complex linear extension of the action preserves the corresponding almost Dirac structure.

Example 4.1.44. Let (M, ω) be a pre-symplectic manifold, let $\Omega^\flat\colon \mathsf{T}M \to \mathsf{T}^*M$ be the associated bundle isomorphism, and let

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\Omega^\sharp \\ \Omega^\flat & 0 \end{pmatrix}$$

be the associated almost GC structure on M. Let G be a Lie group acting smoothly on M. It follows from the discussion in Example 4.1.19 that the G-action on $(M, \mathcal{J}_\omega)$ is canonical if and only if it is symplectic.

Recall that for a smooth action of a compact Lie group G on a manifold M, each connected component of the fixed point set M^G is a closed submanifold of M. (Different components of M^G may have different dimensions.)

Proposition 4.1.45. *Let M be a manifold, and let $\mathcal{J}$ be an almost GC structure on M. Suppose the compact Lie group G acts canonically on $(M, \mathcal{J})$. Then each component of M^G is a split submanifold of $(M, \mathcal{J})$.*

Proof. Let $(M^G)'$ be a component of M^G. First, recall that for each $x \in (M^G)'$ the derivative of the action of G at x defines a linear action of G on T_xM, and that $\mathsf{T}_x(M^G)' = (\mathsf{T}_x M)^G$. Let dg be a bi-invariant Haar measure on G, adjusted so that $dg(G) = 1$. Define a bundle map $\pi\colon (\mathsf{T}M)|_{(M^G)'} \to \mathsf{T}(M^G)'$ by setting

$$\pi_x(v) := \int_G (g \cdot v)\, dg \quad \text{for all } v \in V,$$

for each $x \in (M^G)'$. Define the subbundle $N \subset (\mathsf{T}M)|_{(M^G)'}$ by setting $N_x := \ker \pi_x$ for each $x \in M^G$. That π is a bundle map and N is a vector bundle follow from the naturality of the technique of averaging by integration. By Proposition 4.1.21, for each $x \in V^G$ we know $\mathsf{T}_x M = (\mathsf{T}_x M)^G \oplus N_x = \mathsf{T}_x(M^G)' \oplus N_x$, and that $(\mathsf{T}_x M)^G \oplus \mathrm{Ann}(N_x) = \mathsf{T}_x(M^G)' \oplus \mathrm{Ann}(N_x)$ is preserved by $\mathcal{J}_x$. Thus N is a splitting for $(M^G)' \subset (M, \mathcal{J})$. □

4.2 Background information on G-spaces

In this section we give some brief definitions and results about compact group actions on manifolds which will be required in later sections. The standard reference for the material on equivariant cohomology is [GS99]. The material on orbit spaces and their stratification by orbit types can be found in [DK00, Chapter 2] and [OR04, Chapter 2].

4.2.1 Equivariant cohomology

Let M be a manifold and G be a compact Lie group acting smoothly on M. Consider the space $\Omega^k(M) \otimes S^i(\mathfrak{g}^*)$, where S^i denotes the degree i elements of the symmetric algebra. This is a G-space with action defined by linear extension of the rule $g \cdot (\alpha \otimes p) := ((g^{-1})^*\alpha) \otimes (p \circ \mathrm{Ad}_{g^{-1}})$ for $g \in G$, $\alpha \in \Omega^\star(M)$, $p \in S(\mathfrak{g}^*)$. We can identify $\Omega^k(M) \otimes S^i(\mathfrak{g}^*)$ with the space of degree i polynomial maps $\mathfrak{g} \to \Omega^k(M)$ via

$$\alpha \otimes p \colon \xi \mapsto p(\xi) \cdot \alpha$$

for $\xi \in \mathfrak{g}$. An element of $\Omega^k(M) \otimes S^i(\mathfrak{g}^*)$ is G-invariant if and only if its corresponding polynomial map is G-equivariant with respect to the adjoint action $G \circlearrowright \mathfrak{g}$ and the action $G \circlearrowright \Omega^k(M)$ given by $g \cdot \alpha := (g^{-1})^* \alpha$ for $g \in G$, $\alpha \in \Omega^k(M)$.

Definition 4.2.1. Let M be a manifold and G be a compact Lie group acting smoothly on M. The space of **equivariant differential forms of degree** n on M is

$$\Omega^n_G(M) := \bigoplus_{i=0}^{\lfloor n/2 \rfloor} \left(\Omega^{n-2i}(M) \otimes S^i(\mathfrak{g}^*)\right)^G .$$

The differential $\mathrm{d}_G \colon \Omega^n_G \to \Omega^{n+1}_G$ is defined, viewing equivariant forms as maps $\mathfrak{g} \to \Omega^\star(M)$, by

$$\mathrm{d}_G(\alpha \otimes p)(\xi) := (\mathrm{d}\alpha - \iota_{\xi_M}\alpha)\, p(\xi) \quad \text{for all } \xi \in \mathfrak{g}.$$

The **Cartan model for the G-equivariant cohomology** of M is $H^\bigstar_G(M) := H^\bigstar(\Omega^\bigstar_G, \mathrm{d}_G)$.

Suppose now that G acts freely on M. Then the G-equivariant cohomology of M is naturally isomorphic as a graded algebra to the de Rham cohomology of the quotient M/G,

$$H^\bigstar_G(M) \cong H^\bigstar(M/G).$$

We denote this isomorphism by $\kappa \colon H^\bigstar_G(M) \to H^\bigstar(M/G)$.

Let $B \in \Omega^n(M)$. The form B is called **basic** if it is G-equivariant and if $\iota_{\xi_M} B = 0$ for all $\xi \in \mathfrak{g}$. If there is a differential form $\widetilde{B} \in \Omega^n(M/G)$ such that the pullback of $\widetilde{B}$ by the quotient map $M \to M/G$ equals B, then we say B **descends** to $\widetilde{B}$.

Proposition 4.2.2. *Let M be a manifold and G be a compact Lie group acting smoothly and freely on M.*

(a) If $B \in \Omega^n(M)$ is basic, then B descends to some $\widetilde{B} \in \Omega^n(M/G)$.

(b) If $B \in \Omega^n(M)^G \subset \Omega^n_G(M)$ is equivariantly closed, i.e. $\mathrm{d}_G B = 0$, then B is closed and basic and descends to some closed $\widetilde{B} \in \Omega^n(M/G)$ such that

$$\kappa[B] = [\widetilde{B}],$$

where $[B]$ and $[\widetilde{B}]$ are the cohomology classes of B and $\widetilde{B}$, respectively.

(c) If $\eta \in \Omega^n_G(M)$ is equivariantly closed, then there exists $\Gamma \in \Omega^{n-1}_G(M)$ so that $\eta + \mathrm{d}_G\Gamma \in \Omega^n(M)^G \subset \Omega^n_G(M)$. In this case, since $\eta + \mathrm{d}_G\Gamma$ is equivariantly closed, it descends to some $\widetilde{\eta} \in \Omega^n(M/G)$ such that $\kappa[\eta] = [\widetilde{\eta}]$.

Definition 4.2.3. Let M be a manifold and G be a compact Lie group acting on M smoothly and freely. Then $M \to M/G$ is a (left) principal G-bundle. A **connection** on this bundle is a $\mathfrak{g}$-valued one-form $\theta \in \Omega^1(M, \mathfrak{g})$ such that

(a) θ is G-equivariant, i.e. $g^*\theta = \mathrm{Ad}_g \circ \theta$;

(b) $\theta(\xi_M) \equiv \xi$ for all $\xi \in \mathfrak{g}$.

4.2.2 Orbit type stratification

Let G be a group. For each subgroup H of G, we will denote by (H) the set of subgroups of G that are conjugate to H. Suppose G is a compact Lie group and M be a manifold on which G acts smoothly. Note that the conjugacy relation among subgroups of G preserves closedness, and hence also preserves the property of being a Lie subgroup.

Definition 4.2.4. Let $x \in M$, and let $G_x := \{ g \in G \mid g \cdot x = x \}$ be the isotropy subgroup of x in G. The **orbit type** of the point x, or of $G \cdot x$, is the set (G_x) of subgroups of G that are conjugate to G_x.

Let H be a closed subgroup H of G. The (H)**-orbit type submanifold** of M is the set $M_{(H)} := \{ x \in M \mid G_x \in (H) \}$. The H**-isotropy type submanifold** of M is the set $M_H := \{ x \in M \mid G_x = H \}$. The H**-fixed point submanifold** of M is the set $M^H := \{ x \in M \mid G_x \subset H \}$.

Note that the submanifolds defined above are related by the equation $M_H = M_{(H)} \cap M^H$. Also, two G-orbits in M have the same orbit type if and only if they are G-equivariantly diffeomorphic. This leads one to the following definitions.

Definition 4.2.5. Let $x \in M$, and let $H = G_x$. The **local action type submanifold through** x is the subset $M^{l_x}_{(H)} \subset M$ of points $y \in M$ such that there is a G-equivariant diffeomorphism between G-invariant open neighborhoods of x and y. Define $M^{l_x}_H := M^{l_x}_{(H)} \cap M^H$.

Some important properties of the sets we have defined above are collected in the following proposition. Their proofs can be found in the references cited at the beginning of this section.

Proposition 4.2.6. *Let G be a compact Lie group, and M be a manifold on which G acts smoothly. Let $x \in M$ and put $H = G_x$. Then the following hold.*

(a) $M^{l_x}_{(H)}$, respectively $M^{l_x}_H$, is an open and closed subset of $M_{(H)}$, respectively M_H.

(b) The sets $M^{l_x}_{(H)}$ and $M^{l_x}_H$ are locally closed embedded submanifolds of M, as is each connected component of M_H, of $M_{(H)}$, and of M^H.

(c) $M_H^{l_x}$ *and* $M_{(H)}^{l_X}$ *consists of the union of certain components of* M_H *and* $M_{(H)}$*, respectively.*

(d) M_H *and* $M_H^{l_x}$ *are open in* M^H*.*

(e) $M_{(H)}$ *and* $M_{(H)}^{l_x}$ *are* G*-stable, and* $G \cdot M_H = M_{(H)}$ *and* $G \cdot M_H^{l_x} = M_{(H)}^{l_x}$*.*

(f) Let $N = N_G(H)$ *be the normalizer of* H *in* G*. Both* M_H *and* $M_H^{l_x}$ *are* N*-stable, and* N/H *acts freely on both. Hence* $M_H^{l_x}/N \cong M_H^{l_x}/(N/H)$ *is a manifold.*

(g) The inclusions $M_H \hookrightarrow M_{(H)}$ *and* $M_H^{l_x} \hookrightarrow M_{(H)}^{l_x}$ *induce homeomorphisms* $M_H/N \to M_{(H)}/G$ *and* $M_H^{l_x}/N \to M_{(H)}^{l_x}/G$*. Thus the quotient* $M_{(H)}^{l_x}/G$ *inherits a natural manifold structure.*

(h) Each component of the quotient $M_{(H)}/G$ *inherits a natural manifold structure.*

In general, the orbit space M/G can be a very singular space. It rarely inherits a manifold, or even an orbifold, structure from M. However, because M is the disjoint union of its orbit type submanifolds, we can also partition the orbit space:

$$M/G = \bigsqcup_{(H)} M_{(H)}/G, \tag{4.8}$$

where the disjoint union is taken over all the distinct orbit type submanifolds of M. Since each component of M_H/G is a manifold, we know that, after refining the partition to components, (4.8) is a partition of M/G into manifolds. It is called the **orbit type partition** of M/G.

Remark 4.2.7. All of the above results hold true even if G is an *arbitrary* Lie group, so long as it acts on M both smoothly and *properly*.

4.3 Hamiltonian actions on generalized complex manifolds

In [LT06], the authors proposed the following definition of Hamiltonian actions on GC manifolds.

Definition 4.3.1. Let $(M, \mathcal{J})$ be an untwisted GC manifold, let E be the associated Dirac structure on M, and let G be a Lie group acting canonically on $(M, \mathcal{J})$. This action is **generalized Hamiltonian** if there exists a G-equivariant map $\mu\colon M \to \mathfrak{g}^*$ such that, for all $\xi \in M$,

$$\xi_M = -\mathcal{J}(\mathrm{d}\mu^\xi)$$

or equivalently $\xi_M - i\mathrm{d}\mu^\xi \in \Gamma(E)$. Here $\mu^\xi\colon M \to \mathbb{R}$ is the smooth function defined by $\mu^\xi(x) := \langle \mu(x), \xi \rangle$ for all $x \in M$. The map μ is called a **generalized moment map** for the action $G \circlearrowright (M, \mathcal{J})$.

Let $(M, \mathcal{J}, H)$ be a twisted GC manifold, and let G be a Lie group acting canonically on $(M, \mathcal{J}, H)$. This action is **twisted generalized Hamiltonian** if there exists a G-equivariant map $\mu\colon M \to \mathfrak{g}^*$ and a G-equivariant $\mathfrak{g}^*$-valued one-form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ on M such that, for all $\xi \in M$,

(a) $\xi_M = -\mathcal{J}(\mathrm{d}\mu^\xi) - \alpha^\xi$, (or equivalently $\xi_M + \alpha^\xi - i\mathrm{d}\mu^\xi \in \Gamma(E)$), and

(b) $\iota_{\xi_M} H = \mathrm{d}\alpha^\xi$.

Here μ^ξ is as defined above, and $\alpha^\xi \in \Omega^1(M)$ is the differential one-form on M defined by $(\alpha^\xi)_x(v) := \langle \alpha_x(v), \xi \rangle$ for all $x \in M$, $v \in \mathsf{T}_x M$. The map μ and the one-form α are called a **generalized moment map** and a **moment one-form**, respectively, for the action $G \circlearrowright (M, \mathcal{J}, H)$.

Definition 4.3.2. Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a GK manifold, and let G be a Lie group acting on M and preserving both $\mathcal{J}_1$ and $\mathcal{J}_2$. This action is called **generalized Hamiltonian** if the action of G on $(M, \mathcal{J}_1)$ is generalized Hamiltonian.

Similarly, if $(M, \mathcal{J}_1, \mathcal{J}_2, H)$ be a twisted GK manifold, and the G-action preserves $\mathcal{J}_1$, $\mathcal{J}_2$, and H, then the action is **twisted generalized Hamiltonian** if the action of G on $(M, \mathcal{J}_1, H)$ is twisted generalized Hamiltonian.

Remark 4.3.3.

(a) Note that a moment one-form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ is an equivariant differential form of degree 3, and hence so is $H + \alpha$.

(b) Because E is an isotropic subbundle, the condition that $\xi_M + \alpha^\xi - i\mathrm{d}\mu^\xi \in E$ implies that

$$\begin{aligned} 0 &= \left\langle\!\left\langle \xi_M + \alpha^\xi - i\mathrm{d}\mu^\xi, \xi_M + \alpha^\xi - i\mathrm{d}\mu^\xi \right\rangle\!\right\rangle \\ &= \left\langle\!\left\langle \xi_M + \alpha^\xi, \xi_M + \alpha^\xi \right\rangle\!\right\rangle - 2i \left\langle\!\left\langle \mathrm{d}\mu^\xi, \xi_M + \alpha^\xi \right\rangle\!\right\rangle - \left\langle\!\left\langle \mathrm{d}\mu^\xi, \mathrm{d}\mu^\xi \right\rangle\!\right\rangle \\ &= \iota_{\xi_M}\alpha^\xi - i\,\iota_{\xi_M}\left(\mathrm{d}\mu^\xi\right), \end{aligned}$$

and hence that $\iota_{\xi_M}\alpha^\xi = \iota_{\xi_M}\mathrm{d}\mu^\xi = 0$. Since $\mathrm{d}H = 0$ by assumption, this means that condition (b) in the definition of a twisted generalized Hamiltonian action on a twisted GC manifold is equivalent to requiring that $H + \alpha$ be equivariantly closed:

$$\begin{aligned} \mathrm{d}_G(H + \alpha)(\xi) &= \mathrm{d}H - \iota_{\xi_M}H + \mathrm{d}\alpha^\xi - \iota_{\xi_M}\alpha^\xi \\ &= -\iota_{\xi_M}H + \mathrm{d}\alpha^\xi \end{aligned}$$

for all $\xi \in \mathfrak{g}$.

(c) Given a GC manifold $(M, \mathcal{J})$, one can consider this as a twisted GC manifold $(M, \mathcal{J}, H)$ by setting $H = 0$. Therefore, if a Lie group G acts on $(M, \mathcal{J})$

canonically, we have two notions of whether the action is Hamiltonian. It may be Hamiltonian as an action on $(M, \mathcal{J})$, in which case there is just a moment map, or it may be Hamiltonian as an action on $(M, \mathcal{J}, H)$, in which case there is **both** a moment map **and** a moment one-form. It is potentially interesting to explore both possibilities.

Example 4.3.4. Let (M, ω) be a symplectic manifold, and let G be a Lie group acting on (M, ω) in a Hamiltonian fashion with moment map $\Phi\colon M \to \mathfrak{g}^*$. Recall that this means the G-action is symplectic, the map Φ is G-equivariant, and for all $\xi \in \mathfrak{g}$ we have $\mathrm{d}\Phi^\xi = \iota_{\xi_M}$. Let $\mathcal{J}_\omega$ be the GC structure on M induced by ω. As discussed in Example 3.8 of [LT06], the action of G on $(M, \mathcal{J}_\omega)$ is generalized Hamiltonian, and Φ is a generalized moment map. To see why, observe that for all $\xi \in \mathfrak{g}$ the symplectic moment map condition can be written as $\Omega^\flat(\xi_M) = \mathrm{d}\Phi^\xi$, and so

$$\begin{aligned}
\mathcal{J}_\omega \begin{pmatrix} \xi_M \\ -i\,\mathrm{d}\Phi^\xi \end{pmatrix} &= \begin{pmatrix} 0 & -\Omega^\sharp \\ \Omega^\flat & 0 \end{pmatrix} \begin{pmatrix} \xi_M \\ -i\,\mathrm{d}\Phi^\xi \end{pmatrix} \\
&= \begin{pmatrix} i\,\Omega^\sharp(\mathrm{d}\Phi^\xi) \\ \Omega^\flat(\xi_M) \end{pmatrix} \\
&= \begin{pmatrix} i\,\xi_M \\ \mathrm{d}\Phi^\xi \end{pmatrix} \\
&= i \cdot \begin{pmatrix} \xi_M \\ -i\,\mathrm{d}\Phi^\xi \end{pmatrix},
\end{aligned}$$

i.e. $\xi_M - i\,\mathrm{d}\Phi^\xi \in \Gamma(E_\omega)$.

Theorem 4.3.5. *Let (M, E, H) be a twisted GC manifold, where E is the associated complex Dirac structure, and let G be a Lie group acting on (M, E, H) in a Hamiltonian fashion with moment map $\mu\colon M \to \mathfrak{g}^*$ and moment one-form*

$\alpha \in \Omega^1(M, \mathfrak{g}^*)$. If $j\colon S \hookrightarrow M$ is a G-stable twisted GC submanifold of (M, E, H), then the restriction of the action of G to (S, E_S, j^*H) is Hamiltonian with moment map $\mu|_S\colon S \to \mathfrak{g}^*$ and moment one-form $j^*\alpha \in \Omega^1(S, \mathfrak{g}^*)$.

Proof. First we will prove that the action of G on S preserves E_S. Let $x \in S$ and $(X, \lambda) \in (\mathsf{T}_{\mathbb{C},x}S \oplus \mathsf{T}^*_{\mathbb{C},x}M) \cap E_x$, which means that $(X, \lambda|_S) \in E_{S,x}$. Then for any $g \in G$ we have

$$g \cdot (X + j^*\lambda) = g_*(X) + (g^{-1})^*\lambda|_S.$$

Because S is G-stable, the inclusion $j\colon S \hookrightarrow M$ is G-equivariant, i.e. the G-action commutes with j. Hence $j_*\colon \mathsf{T}S \hookrightarrow \mathsf{T}M$ is G-equivariant, so $g_*(X) \in \mathsf{T}_{g\cdot x}S$. Also

$$(g^{-1})^*\left(\lambda|_S\right) = (g^{-1})^* j^*\lambda = j^*(g^{-1})^*\lambda = \left((g^{-1})^*\lambda\right)|_S.$$

Since E is G-stable, we have $g \cdot (X + \lambda) = g_*(X) + (g^{-1})^*\lambda \in E_{g\cdot x}$. Therefore $g \cdot (X, j^*\lambda) = (g_*(X), j^*(g^{-1})^*\lambda) \in E_{S,g\cdot x}$. Thus E_S is G-stable.

Now suppose that (S, E_S, j^*H) is a GC submanifold of (M, E, H), meaning that E_S is a vector bundle, that $E_S \cap \overline{E_S} = 0$, and that E_S is j^*H-twisted Courant involutive. Since j is G-equivariant, for all $\xi \in \mathfrak{g}$ we have $\xi_M|_S = \xi_S$, $(j^*\mu)^\xi = j^*(\mu^\xi)$, and $(j^*\alpha)^\xi = j^*(\alpha^\xi)$. Furthermore, by the naturality of the exterior derivative we have

$$\mathrm{d}(j^*\mu)^\xi = \mathrm{d}j^*(\mu^\xi) = j^*(\mathrm{d}\mu^\xi),$$

so $\mathrm{d}(\mu|_S)^\xi = (\mathrm{d}\mu^\xi)|_S$. For each $x \in S \subset M$, since $\left(\xi_M + \alpha^\xi - i\,\mathrm{d}\mu^\xi\right)\big|_x \in E_x$, this means that

$$\left(\xi_S + \alpha^\xi|_S - i\,\mathrm{d}\mu^\xi|_S\right)\big|_x \in E_{S,x}.$$

Again using the G-equivariance of j, for all $x \in S$ we have

$$\iota_{\xi_S} j^*(H)\big|_x = j^*(\iota_{\xi_S} H)\big|_x = j^*(\iota_{\xi_M} H)\big|_x = j^*(\mathrm{d}\alpha^\xi)\big|_x = \mathrm{d}(j^*\alpha)^\xi\big|_x.$$

Thus the action of G on (S, E_S, j^*H) is twisted Hamiltonian with moment map $\mu|_S$ and moment one-form $\alpha|_S$. $\square$

The above result holds also for the untwisted case, of course, by putting $H = 0$ and $\alpha = 0$.

The following three results are exactly what makes reduction of generalized Hamiltonian manifolds possible.

Theorem 4.3.6 (Lemma 3.8 and Proposition 4.6 of [LT06])**.** *Let a compact Lie group G act on a GC manifold $(M, \mathcal{J})$, respectfully a GK manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$, in a Hamiltonian fashion with moment map $\mu\colon M \to \mathfrak{g}^*$. Suppose $a \in \mathfrak{g}^*$ is an element such that G acts freely on the inverse image $\mu^{-1}(\mathcal{O}_a)$ of the coadjoint orbit $\mathcal{O}_a := \mathrm{Coad}_G(a)$ of G through a. Then the quotient space $\mu^{-1}(\mathcal{O}_a)/G$ inherits a natural GC structure $\widetilde{\mathcal{J}}$ from $\mathcal{J}$, respectfully a natural GK structure $(\widetilde{\mathcal{J}}_1, \widetilde{\mathcal{J}}_2)$ from $(\mathcal{J}_1, \mathcal{J}_2)$.*

Lemma 4.3.7 (Lemma A.6 of [LT06])**.** *Let a compact Lie group G act freely on a manifold M. Let H be a G-invariant and closed three-form, and let $\alpha\colon \mathfrak{g} \to \Omega^1(M)$ be an equivariant map. Fix a connection $\theta \in \Omega^1(M, \mathfrak{g})$ on the principal G-bundle $M \to M/G$. Then if $H + \alpha \in \Omega^3_G(M)$ is equivariantly closed, there exists a natural form $\Gamma \in \Omega^2(M)^G$ so that $\iota_{\xi_M}\Gamma = \alpha^\xi$ for all $\xi \in \mathfrak{g}$. Thus $H + \alpha + \mathrm{d}_G\Gamma \in \Omega^3(M)^G \subset \Omega^3_G(M)$ is closed and basic and so descends to a closed form $\widetilde{H} \in \Omega^3(M/G)$ so that $[\widetilde{H}] = \kappa[H + \alpha]$.*

Theorem 4.3.8 (Propositions A.7 and A.10 of [LT06])**.** *Let a compact Lie group G act on a twisted GC manifold $(M, \mathcal{J}, H)$, respectfully a twisted GK manifold $(M, \mathcal{J}_1 m \mathcal{J}_2, H)$, in a Hamiltonian fashion with moment map $\mu\colon M \to \mathfrak{g}^*$ and moment one-form $\alpha \in \Omega^1(M, \mathfrak{g})$. Suppose $a \in \mathfrak{g}^*$ is an element such that G acts freely* *on $\mu^{-1}(\mathcal{O}_a)$. Assume that $H+\alpha$ is equivariantly closed. Given a connection on the principal G-bundle $\mu^{-1}(\mathcal{O}_a) \to \mu^{-1}(\mathcal{O}_a)/G$, the quotient space $\mu^{-1}(\mathcal{O}_a)/G$ inherits an $\widetilde{H}$-twisted GC structure $\widetilde{\mathcal{J}}$ from $\mathcal{J}$, respectfully an $\widetilde{H}$-twisted GK structure $(\widetilde{\mathcal{J}}_1, \widetilde{\mathcal{J}}_2)$ from $(\mathcal{J}_1, \mathcal{J}_2)$, where $\widetilde{H}$ is defined as in Lemma 4.3.7 above. Up to B-transform, these inherited structures are independent of our choice of connection.*

Definition 4.3.9. The quotient space $\mu^{-1}(\mathcal{O}_a)/G$ in Theorems 4.3.6 and 4.3.8 is called the **generalized complex quotient** (or **generalized Kähler quotient**, as applicable), or the **Lin–Tolman quotient**, of M by G at level a. We use the notation

$$M_a := \mu^{-1}(\mathcal{O}_a)/G.$$

Remark 4.3.10. As noted in Example 3.9 of [LT06], in the context of the hypotheses of Theorem 4.3.6, if the GC structure and moment map come from a *symplectic* structure and moment map, then the GC structure on the quotient is exactly the one induced by the Marsden–Weinstein ssymplectic structure on the quotient.

The following result will be useful to us later. Its proof follows trivially from the definitions of generalized and twisted generalized Hamiltonian actions.

Lemma 4.3.11. *Let $(M, \mathcal{J}, H)$ be a twisted GC manifold with a Hamiltonian action of a Lie group G, and let $\mu\colon M \to \mathfrak{g}^*$ and $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ be a moment map and moment one-form, respectively. Let $K \subset G$ be a Lie subgroup. Then the induced action of K on $(M, \mathcal{J}, H)$ is also Hamiltonian, with generalized moment map and moment one-form the compositions of μ and α, respectively, with the projection $\mathfrak{g}^* \twoheadrightarrow \mathfrak{k}^*$ dual to the inclusion $\mathfrak{k} \hookrightarrow \mathfrak{g}$:*

$$M \xrightarrow{\mu} \mathfrak{g}^* \twoheadrightarrow \mathfrak{k}^*, \qquad \mathsf{T}M \xrightarrow{\alpha} \mathfrak{g}^* \twoheadrightarrow \mathfrak{k}^*.$$

Example 4.3.12. Let G be a Lie group, and let $(M_i, \mathcal{J}_i, H_i)$ be a twisted GC manifold on which G acts in a Hamiltonian fashion with moment map $\mu_i\colon M_i \to \mathfrak{g}^*$ and moment one-form $\alpha_i \in \Omega^1(M_i, \mathfrak{g}^*)$, for $i = 1, 2$. Let $(M_1 \times M_2, \mathcal{J}, H)$ be the product of these two GC manifolds, as defined in Example 4.1.36. Recall that $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2)$ and $H = \pi_1^* H_1 + \pi_2^* H_2$, where $\pi_i\colon M_1 \times M_2 \to M_i$ is the natural projection for $i = 1, 2$. Define by $\mu\colon M_1 \times M_2 \to \mathfrak{g}^* \oplus \mathfrak{g}^*$ and $\alpha \in \Omega^1(M_1 \times M_2, \mathfrak{g}^* \oplus \mathfrak{g}^*)$ by $\mu = \pi_1^*\mu_1 + \pi_2^*\mu_2$ and $\alpha = \pi_1^*\alpha_1 + \pi_2^*\alpha_2$. It is easy to check that the action of $G \times G$ on $M_1 \times M_2$ is twisted generalized Hamiltonian with moment map μ and moment one-form α.

Embedding G diagonally in $G \times G$, we obtain a Hamiltonian action of G on $M_1 \times M_2$. The projection $\mathfrak{g}^* \oplus \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}^*$ induced by this embedding is given by addition: $(\lambda_1, \lambda_2) \mapsto \lambda_1 + \lambda_2$, so a moment map and moment one-form for the G-action on $M_1 \times M_2$ is given by

$$M_1 \times M_2 \to \mathfrak{g}^*, \quad (x_1, x_2) \mapsto \mu_1(x_1) + \mu_2(x_2)$$

and

$$\mathsf{T}M_1 \times \mathsf{T}M_2 \to \mathfrak{g}^*, \quad (X_1, X_2) \mapsto \alpha_1(X_1) + \alpha_2(X_2),$$

respectively.

Perhaps the most important instance of the construction of Example 4.3.12 is if we start with an arbitrary twisted generalized Hamiltonian G-manifold, $(M, \mathcal{J}, H, \mu, \alpha)$, and let the second GC manifold be a coadjoint orbit $\mathcal{O}_a := \mathrm{Coad}_G(a)$ in $\mathfrak{g}^*$, where $a \in \mathfrak{g}^*$ is some fixed element. Let ω_a be the canonical symplectic structure on $\mathcal{O}_a$. The action of G on $\mathcal{O}_a$ is Hamiltonian in the symplectic sense, with moment map given by the inclusion $\mathcal{O}_a \hookrightarrow \mathfrak{g}^*$. Using the symplectic structure $-\omega_a$ instead, the action is still Hamiltonian, but now the moment map is given by the negative inclusion $\mathcal{O}_a \to \mathfrak{g}^*$, $\lambda \mapsto -\lambda$.

As described in Examples 4.1.33 and 4.3.4, the symplectic structure $-\omega_a$ induces a GC structure $\mathcal{J}_a$ on $\mathcal{O}_a$, and the G-action on $\mathcal{O}_a$ is generalized Hamiltonian with the same moment map. Viewing $(\mathcal{O}_a, \mathcal{J}_a)$ as a twisted GC manifold where the twisting is by the zero three-form, the G-action is twisted generalized Hamiltonian with a constantly vanishing moment one-form. Then the diagonal G-action on $M \times \mathcal{O}_a$ is twisted generalized Hamiltonian with moment map

$$\mu' \colon M \times \mathcal{O}_a \to \mathfrak{g}^*, \quad (x, \lambda) \mapsto \mu(x) - \lambda$$

and moment one-form

$$\alpha' \colon \mathsf{T}M \times \mathsf{T}(\mathcal{O}_a) \to \mathfrak{g}^*, \quad (X, Y) \mapsto \alpha(X).$$

The reason this construction is important is that it is the basis of the **shifting trick**. If one wishes to reduce the M by G at level $a \in \mathfrak{g}^*$, one can instead consider the reduction of $M \times \mathcal{O}_a$ by G at level 0, because

$$M_a \approx (M \times \mathcal{O}_a)_0$$

as topological spaces. To see this, observe that $\mu^{-1}(\mathcal{O}_a)$ and $(\mu')^{-1}(0)$ are G-equivariantly homeomorphic via the maps

$$\mu^{-1}(\mathcal{O}_a) \to (\mu')^{-1}(0), \quad x \mapsto (x, \mu(x))$$

and

$$(\mu')^{-1}(0) \to \mu^{-1}(\mathcal{O}_a), \quad (x, \lambda) \mapsto x.$$

4.4 Partition of the generalized reduced space

Let M be a manifold, G be a Lie group acting on M smoothly, and $\mu \colon M \to \mathfrak{g}^*$ a smooth, G-equivariant map. Let $a \in \mathfrak{g}^*$. By equivariance the pre-image $\mu^{-1}(\mathcal{O}_a)$

of the coadjoint orbit $\mathcal{O}_a$ is preserved by G, and so we can consider the quotient space $\mu^{-1}(\mathcal{O}_a)/G$. Let $M = \bigsqcup M_{(H)}$ be the orbit type partition of M. Because each set $M_{(H)}$ is stable under G, each intersection $\mu^{-1}(\mathcal{O}_a) \cap M_{(H)}$ is also stable under G, so the orbit type partition of M descends to a partition

$$\mu^{-1}(\mathcal{O}_a)/G = \bigsqcup_{(H)} \left(\mu^{-1}(\mathcal{O}_a) \cap M_{(H)}\right)/G$$

of the quotient $\mu^{-1}(\mathcal{O}_a)/G$.

Suppose now M is a symplectic manifold, the G-action is Hamiltonian, and μ is a moment map. In this case the quotient space $M_a := \mu^{-1}(\mathcal{O}_a)/G$ is called the **symplectic reduction**, or **Marsden–Weinstein quotient**, of M at level a. The symplectic moment map condition is that $\mathrm{d}\mu^\xi = \iota_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$. If G acts freely on $\mu^{-1}(\mathcal{O}_a)$, then each ξ_M is nonzero on $\mu^{-1}(\mathcal{O}_a)$, which by the non-degeneracy of ω implies that a is a regular value of μ. Therefore $\mu^{-1}(\mathcal{O}_a) \subset M$ is a submanifold, so M_a is a manifold. In this case, Marsden and Weinstein proved that M_a inherits a natural symplectic structure. Theorems 4.3.6 and 4.3.8, proved in [LT06], are analogues of this result.

In the event that the symplectic quotient is singular, one can consider the individual parts of the partitioned quotient. In [SL91], Lerman and Sjamaar proved that each component of $(M_a)_{(H)} := \left(\mu^{-1}(\mathcal{O}_a) \cap M_{(H)}\right)/G$ inherits a natural symplectic structure. The main results of this paper are analogues of this in the generalized complex case.

Remark 4.4.1. By the symplectic moment map condition, $\mathrm{d}\mu^\xi = \iota_{\xi_M}\omega$, if $a \in \mathfrak{g}^*$ is a regular value of μ, then each vector field ξ_M is nowhere zero on $\mu^{-1}(a)$. This means that the action of G on $\mu^{-1}(a)$ is at least *locally free*, which means that the quotient M_a is at worst an orbifold, to which Marsden and Weinstein were able to associate a symplectic structure. By Sard's Theorem, a generic value of μ will be regular, so the generic result of symplectic reduction is a symplectic orbifold.

If $(M, \mathcal{J})$ is an untwisted GC manifold with moment map μ, then the generalized moment map condition, $\mathcal{J}(\mathrm{d}\mu^\xi) = -\xi_M$, likewise guarantees the equivalence of regular values and local freeness of the action, so M_a is at worst an orbifold. However, if $(M, \mathcal{J}, H)$ is a twisted GC manifold with moment map μ and moment one-form α, then this equivalence may no longer hold, due to the presence of the moment one-form in the moment condition:

$$\mathcal{J}(\mathrm{d}\mu^\xi) = -\xi_M - \alpha^\xi.$$

Specifically, ξ_M could vanish even if $\mathcal{J}(\mathrm{d}\mu^\xi)$ does not. Therefore, it seems that the generic result of GC reduction may be a GC singular space.

Before stating and proving our main theorem, we need several lemmas.

Lemma 4.4.2. *Let $(M, \mathcal{J}, H)$ be a* compact*, twisted GC manifold, and let G be a compact Lie group acting on $(M, \mathcal{J}, H)$ in a Hamiltonian fashion with moment map $\mu\colon M \to \mathfrak{g}^*$ and moment one-form $\alpha \in \Omega^1(M, \mathfrak{g})$. If the G-action on M is trivial, then* $\mathrm{d}\mu = \alpha \equiv 0$.

In Lemma 5.5 of [BL08], the authors proved the above result in the case that G is a torus; however, their proof holds verbatim in the non-abelian, but still compact, case. It relies on viewing the components of μ, (μ^ξ for $\xi \in \mathfrak{g}$), as the real parts of pseudo-holomorphic functions (with respect to an almost complex structure derived from the GC structure) and applying a version of the Maximum Principle, a course first taken in [Nit09]. At the present time, there does not appear to be a fully satisfactory proof of this version of the Maximum Principle, so we provide one here.

Definition 4.4.3. Let M be a $2n$-dimensional manifold, and $J \colon \mathsf{T}M \to \mathsf{T}M$ be an almost complex structure on M. If $f, g \in C^\infty(M)$, then the smooth function $f + i\,g \colon M \to \mathbb{C}$ is called **psuedo-holomorphic** with respect to J, or **J-holomorphic**, if it satisfies

$$(\mathrm{d}f + i\,\mathrm{d}g) \circ J = i\,(\mathrm{d}f + i\,\mathrm{d}g). \tag{4.9}$$

For the setup to stating and proving the Maximum Principle for the real part of a pseudo-holomorphic function, we follow [BL08, Appendix A].

Let $x_1, \ldots, x_{2n} \in C^\infty(U)$, $U \subset M$, be local coordinates on an almost complex manifold (M, J), and let $(J_{pk})_{p,k=1}^{2n}$ be the matrix for J in these coordinates, so that

$$J\left(\frac{\partial}{\partial x_k}\right) = \sum_{p=1}^{2n} J_{pk}\,\frac{\partial}{\partial x_p}$$

for $k = 1, \ldots, n$. Suppose $f, g \in C^\infty(M)$. Separating into real and imaginary parts, we see that (4.9) is equivalent to the conditions

$$\mathrm{d}f \circ J = -\mathrm{d}g \quad \text{and} \quad \mathrm{d}g \circ J = \mathrm{d}f.$$

Since $\left\langle \mathrm{d}f, \frac{\partial}{\partial x_k} \right\rangle = \frac{\partial f}{\partial x_k}$, we know $\mathrm{d}f \circ J = -\mathrm{d}g$ if and only if

$$\begin{aligned}
-\frac{\partial g}{\partial x_k} &= \left\langle -\mathrm{d}g, \frac{\partial}{\partial x_k} \right\rangle \\
&= \left\langle \mathrm{d}f, J\left(\frac{\partial}{\partial x_k}\right) \right\rangle \\
&= \left\langle \mathrm{d}f, \sum_{p=1}^{2n} J_{pk}\,\frac{\partial}{\partial x_p} \right\rangle \\
&= \sum_{p=1}^{2n} J_{pk} \left\langle \mathrm{d}f, \frac{\partial}{\partial x_p} \right\rangle \\
&= \sum_{p=1}^{2n} J_{pk}\,\frac{\partial f}{\partial x_p}
\end{aligned}$$

for all $k = 1, \ldots, n$, and $\mathrm{d}g \circ J = \mathrm{d}f$ if and only if

$$\begin{aligned}
\frac{\partial f}{\partial x_k} &= \left\langle \mathrm{d}f, \frac{\partial}{\partial x_k} \right\rangle \\
&= \left\langle \mathrm{d}g, J\left(\frac{\partial}{\partial x_k}\right) \right\rangle \\
&= \left\langle \mathrm{d}g, \sum_{p=1}^{2n} J_{pk} \frac{\partial}{\partial x_p} \right\rangle \\
&= \sum_{p=1}^{2n} J_{pk} \left\langle \mathrm{d}g, \frac{\partial}{\partial x_p} \right\rangle \\
&= \sum_{p=1}^{2n} J_{pk} \frac{\partial g}{\partial x_p}
\end{aligned}$$

for all $k = 1, \ldots, n$. Therefore (4.9) is equivalent to the **Cauchy–Riemann Equations**,

$$\begin{cases} \dfrac{\partial f}{\partial x_k} = \displaystyle\sum_{p=1}^{2n} J_{pk} \frac{\partial g}{\partial x_p}, \\ \dfrac{\partial g}{\partial x_k} = -\displaystyle\sum_{p=1}^{2n} J_{pk} \frac{\partial f}{\partial x_p} \end{cases} \qquad \text{for } k = 1, \ldots, n.$$

Therefore, if $f + ig$ is indeed J-holomorphic, then

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_k^2} &= \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_k}\right) \\
&= \frac{\partial}{\partial x_k}\left(\sum_{p=1}^{2n} J_{pk}\,\frac{\partial g}{\partial x_p}\right) \\
&= \sum_{p=1}^{2n}\left[\frac{\partial J_{pk}}{\partial x_k}\,\frac{\partial g}{\partial x_p} + J_{pk}\,\frac{\partial}{\partial x_k}\left(\frac{\partial g}{\partial x_p}\right)\right] \\
&= \sum_{p=1}^{2n}\left[\frac{\partial J_{pk}}{\partial x_k}\,\frac{\partial g}{\partial x_p} + J_{pk}\,\frac{\partial}{\partial x_p}\left(\frac{\partial g}{\partial x_k}\right)\right] \\
&= \sum_{p}^{2n}\left[-\frac{\partial J_{pk}}{\partial x_k}\cdot\left(\sum_{q=1}^{2n} J_{qp}\,\frac{\partial f}{\partial x_q}\right) - J_{pk}\,\frac{\partial}{\partial x_p}\,\left(\sum_{q=1}^{2n} J_{qk}\,\frac{\partial f}{\partial x_q}\right)\right] \\
&= \sum_{p=1}^{2n}\left[-\frac{\partial J_{pk}}{\partial x_k}\cdot\left(\sum_{q=1}^{2n} J_{qp}\,\frac{\partial f}{\partial x_q}\right) - J_{pk}\cdot\sum_{q=1}^{2n}\left(\frac{\partial J_{qk}}{\partial x_p}\,\frac{\partial f}{\partial x_q} + J_{qk}\,\frac{\partial^2 f}{\partial x_p\,\partial x_q}\right)\right] \\
&= -\sum_{p,q=1}^{2n}\left(J_{qp}\,\frac{\partial J_{pk}}{\partial x_k}\,\frac{\partial f}{\partial x_q} + J_{pk}\,\frac{\partial J_{qk}}{\partial x_p}\,\frac{\partial f}{\partial x_q} + J_{pk}\,J_{qk}\,\frac{\partial^2 f}{\partial x_p\,\partial x_q}\right),
\end{aligned}$$

and therefore

$$\frac{\partial^2 f}{\partial x_k^2} + \sum_{p,q=1}^{2n}\left(J_{pk}\,J_{qk}\,\frac{\partial^2 f}{\partial x_p\,\partial x_q}\right) + \sum_{p,q=1}^{2n}\left(J_{qp}\,\frac{\partial J_{pk}}{\partial x_k} + J_{pk}\,\frac{\partial J_{qk}}{\partial x_p}\right)\,\frac{\partial f}{\partial x_q} = 0. \tag{4.10}$$

Let L denote the second-order linear differential operator

$$L := \sum_{k=1}^{2n}\left[\frac{\partial^2}{\partial x_k^2} + \sum_{p,q=1}^{2n}\left(J_{pk}\,J_{qk}\,\frac{\partial^2}{\partial x_p\,\partial x_q}\right) + \sum_{p,q=1}^{2n}\left(J_{qp}\,\frac{\partial J_{pk}}{\partial x_k} + J_{pk}\,\frac{\partial J_{qk}}{\partial x_p}\right)\,\frac{\partial}{\partial x_q}\right]. \tag{4.11}$$

The lead coefficient matrix of L has (p,q)-entry $a_{pq} = \delta_{pq} + \sum_{k=1}^{2n} J_{pk}J_{qk}$, where δ is the Kronecker delta.

Suppose now that $x \in U \subset M$ is a point where the almost complex structure J has standard form with respect to these local coordinates:

$$J(x) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then it is easy to check that $\sum_{k=1}^{2n} J_{pk}(x)\, J_{qk}(x) = \delta_{pq}$, so that $a_{pq}(x) = 2\delta_{pq}$ and the lead coefficient matrix of L is $2I_n$, which is a positive-definite symmetric matrix. This means precisely that L is **elliptic** at x. Furthermore, by shrinking the domain U of our chart if necessary, we can assume that the lead coefficient matrix of L is positive-definite, and hence L is elliptic, throughout U. For all $y \in U$, let $\lambda(y)$ and $\Lambda(y)$ be the smallest and largest eigenvalues of the positive-definite symmetric matrix $[a_{pq}(y)]_{p,q=1}^{2n}$, and note that $\lambda, \Lambda \colon U \to \mathbb{R}$ are continuous functions. Then since $\Lambda(x)/\lambda(x) = 1$, by shrinking U further, if necessary, we can assume that Λ/λ is bounded in U, which means that L is **uniformly elliptic** in U. (For the relevant definitions, see the introduction to Chapter 3 in [GT01].)

We are now prepared to prove the Maximum Principal for the real part of a pseudo-holomorphic function.

Theorem 4.4.4. *Let (M, J) be a compact almost-complex manifold of dimension $2n$, and let $f, g \in C^\infty(M)$. If $f + i\,g$ is J-holomorphic, then f is a constant function.*

Proof. Suppose $f + i\,g$ is J-holomorphic. Since M is compact and f is continuous, f must attain an absolute minimum and an absolute maximum on M. Suppose these values are attained at p_0 and p_1 in M, respectively. We will prove that $f(p_0) = f(p_1)$, which implies that f is constant.

Let $c\colon [0, 1] \to M$ be a continuous curve in M from p_0 to p_1. For each point p on this curve, choose coordinates $x_1, \ldots, x_{2n} \colon U_p \to M$ such that, with respect to these coordinates, $J(p)$ has matrix

$$J(p) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and the differential operator L defined by (4.11) is uniformly elliptic on U_p. By (4.10), we know that $Lf = 0$ on U. Therefore, by the Strong Maximum Principle for elliptic differential operators, [GT01, Theorem 3.5], it follows that f is constant on U_p. Since the image of the path c is compact, we know that we can cover it with a finite number of these open sets U_p, each of which must overlap with another. It follows that f is constant on the image of the path c, and hence its values $f(p_0)$ and $f(p_1)$ at the path's endpoints are the same. This completes the proof. $\square$

Lemma 4.4.5. *Let G be a compact Lie group, let $K \subset G$ be a closed Lie subgroup, and let $N = N_G(K)$ be the normalizer of K in G. Then the projection $\mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^*$ dual to the inclusion $\mathfrak{n} \hookrightarrow \mathfrak{g}$ induces an isomorphism*

$$\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k}) \cap (\mathfrak{g}^*)^K \cong \mathrm{Ann}_{\mathfrak{n}^*}(\mathfrak{k}).$$

Proof. Consider the induced left action of K on the quotient G/K. Let $g \in G$, and denote by $[g]$ its image in the quotient G/K. Note that

$$\begin{aligned} [g] \in (G/K)^K \quad &\iff \quad [kg] = [g] \text{ for all } k \in K \\ &\iff \quad g^{-1}kg \in K \text{ for all } k \in K \\ &\iff \quad g^{-1} \in N \\ &\iff \quad g \in N. \end{aligned}$$

Therefore $N/K = (G/K)^K$, and it follows that $\mathfrak{n}/\mathfrak{k} = (\mathfrak{g}/\mathfrak{k})^K$. Since one can construct a K-equivariant linear isomorphism $\mathfrak{g}/\mathfrak{k} \to (\mathfrak{g}/\mathfrak{k})^*$, which gives an isomorphism between their K-fixed sets, we have

$$\dim \mathrm{Ann}_{\mathfrak{n}^*}(\mathfrak{k}) = \dim(\mathfrak{n}/\mathfrak{k})^* = \dim \left((\mathfrak{g}/\mathfrak{k})^K\right)^* = \dim \left((\mathfrak{g}/\mathfrak{k})^*\right)^K = \dim \left(\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k})\right)^K.$$

Now, let $p\colon \mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^*$ be the projection dual to the inclusion $\mathfrak{n} \hookrightarrow \mathfrak{g}$. Clearly $p\left((\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k}))^K\right) \subset \mathrm{Ann}_{\mathfrak{n}^*}(\mathfrak{k})$. Since these two sets have the same dimension, to show that p induces an isomorphism between them, it suffices to show that this set containment is actually an equality.

Because K is closed in G it is also compact, so we can find a K-stable linear subspace $\mathfrak{a} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}$ as vector spaces. (For instance, choose a K-invariant inner product on $\mathfrak{g}$ and let $\mathfrak{a} = \mathfrak{n}^{\perp}$.) Let $\lambda \in \mathrm{Ann}_{\mathfrak{n}^*}(\mathfrak{k})$. Define $\widetilde{\lambda} \in \mathfrak{g}^*$ by putting $\widetilde{\lambda}(X) = \lambda(v)$ if $X \in \mathfrak{n}$ and $\widetilde{\lambda}(X) = 0$ if $X \in \mathfrak{a}$. Certainly $\widetilde{\lambda}$ vanishes on $\mathfrak{k}$. To prove that $\widetilde{\lambda}$ is K-invariant, let $k \in K$ and let $X \in \mathfrak{g}$. Write $X = Y + Z$ for $Y \in \mathfrak{a}$ and $Z \in \mathfrak{n}$. Since $\mathfrak{a}$ is K-stable, we have $k^{-1} \cdot Y \in \mathfrak{a}$ and so $\widetilde{\lambda}(k^{-1} \cdot Y) = 0$. Because $\mathfrak{n}/\mathfrak{k} \subset (\mathfrak{g}/\mathfrak{k})^K$, we know that $k^{-1} \cdot Z$ and Z are the same up to an element of $\mathfrak{k}$, so there exists some $W \in \mathfrak{k}$ such that $k^{-1} \cdot Z = Z + W$. Since $\mathfrak{n}$ is K-stable, we know $k^{-1} \cdot Z \in \mathfrak{n}$. Together with the fact that λ vanishes on $\mathfrak{k}$, this implies that

$$\widetilde{\lambda}(k^{-1} \cdot Z) = \lambda(k^{-1} \cdot Z) = \lambda(Z + W) = \lambda(Z) + \lambda(W) = \lambda(Z).$$

Therefore

$$\begin{aligned}(k \cdot \widetilde{\lambda})(X) &= \widetilde{\lambda}(k^{-1} \cdot X) = \widetilde{\lambda}(k^{-1} \cdot Y + k^{-1} \cdot Z) \\ &= \widetilde{\lambda}(k^{-1} \cdot Y) + \widetilde{\lambda}(k^{-1} \cdot Z) \\ &= \lambda(Z) = \widetilde{\lambda}(Y + Z) = \widetilde{\lambda}(X).\end{aligned}$$

Hence $\widetilde{\lambda} \in (\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k}))^K$. Thus $p\left((\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k}))^K\right) = \mathrm{Ann}_{\mathfrak{n}^*}(\mathfrak{k})$. □

Theorem 4.4.6 (Singular generalized reduction)**.**

(a) Let $(M, \mathcal{J})$ be a GC manifold, and let G be a compact group acting in a Hamiltonian fashion on $(M, \mathcal{J})$ with generalized moment map $\mu \colon M \to \mathfrak{g}^*$. Let $a \in \mathfrak{g}^*$, and let $M_a = \bigsqcup (M_a)_{(H)}$ be the orbit type partition of the GC quotient of $(M, \mathcal{J})$ by G at level a. Then each component of each $(M_a)_{(H)}$ inherits a natural GC structure from $(M, \mathcal{J})$.

(b) *Let $(M, \mathcal{J}, H)$ be a* compact *GC manifold, and let G be a compact group acting in a Hamiltonian fashion on $(M, \mathcal{J}, H)$ with generalized moment map $\mu\colon M \to \mathfrak{g}^*$ and moment one-form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$. Assume that $H + \alpha$ is equivariantly closed. Let $a \in \mathfrak{g}^*$, and $M_a = \bigsqcup (M_a)_{(H)}$ be the orbit type partition of the GC quotient of $(M, \mathcal{J}, H)$ by G at level a. Then each component of each $(M_a)_{(H)}$ inherits a twisted GC structure from $(M, \mathcal{J}, H)$, natural up to B-transform.*

Proof. We begin with the twisted case.

First we prove the theorem in the case that $a = 0$.

Let $x \in M$ and $K = G_x$. Note that this implies that K is a closed subgroup of G, and is hence compact. Clearly K acts canonically on $(M, \mathcal{J}, H)$, since G does. By part (c) of Proposition 4.2.6, $M_K^{l_x}$ is open in M^K. It follows that every component of M^K intersecting $M_K^{l_x}$ has the same dimension as $M_K^{l_x}$. Let M_x^K be the union of components of M^K having nontrivial intersection with $M_K^{l_x}$. Since each component of M^K is a manifold, it follows that M_x^K is also . Furthermore, by Proposition 4.1.45 we know that each connected component of M^K is a split submanifold of $(M, \mathcal{J})$, and hence also a twisted GC submanifold. Therefore so is M_x^K.

Let Z_K^x be the union of components of $M_K^{l_x}$ that have nontrivial intersection with $\mu^{-1}(0)$. Since Z_K^x is open in $M_K^{l_x}$, which is open in M_x^K, as discussed in Remark 4.1.39 we know that Z_K^x is a twisted GC submanifold of M_x^K, and hence also of M. Let $j\colon Z_K^x \hookrightarrow M$ be the inclusion, and denote the (j^*H)-twisted GC structure of Z_K^x by $\mathcal{J}'$,

Let $N = N_G(K)$ be the normalizer of K in G. By part (e) of Proposition 4.2.6, we know $M_K^{l_x}$ is N-stable. In fact, so is Z_K^x, as we now show. Note that connected components of manifolds are path-connected. Therefore, for any $n \in N$, if $y, z \in M_K^{l_x}$ are in the same component, then so are $n \cdot y$ and $n \cdot z$. Now let $n \in N$ and $y \in Z_K^x$. By the definition of Z_K^x, there exists some z in the same component of $M_K^{l_x}$ as y such that $z \in \mu^{-1}(0)$. Since $\mu^{-1}(0)$ is N-stable, this means $n \cdot z \in \mu^{-1}(0) \cap M_K^{l_x} \subset Z_K^x$, and hence $n \cdot y \in Z_K^x$ as well.

Now we will show that $\mu(Z_K^x)$ and $\alpha(\mathsf{T} Z_K^x)$ lie in $\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k}) \cap (\mathfrak{g}^*)^K$, where $\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k})$ denotes the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$. Since $(M^K)'$ is fixed point-wise by K and μ and α are equivariant, we these two sets are contained in $(\mathfrak{g}^*)^K$. Because $(M^K)'$ is closed in M, it is compact. Since K acts trivially on $(M^K)'$, it follows from Theorem 4.3.5 and Lemmas 4.3.11 and 4.4.2 that $\mathrm{d}\mu^\xi = \alpha^\xi = 0$ on $\mathsf{T}(M^K)'$, and hence on $\mathsf{T} Z_K^x$, for all $\xi \in \mathfrak{k}$. Hence μ^ξ is locally constant on Z_K^x for all $\xi \in \mathfrak{k}$. Because each component of Z_K^x has nonempty intersection with $\mu^{-1}(0)$, it follows that $\mu^\xi = 0$ on Z_K^x for all $\xi \in \mathfrak{k}$, so $\mu(Z_K^x) \subset \mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k})$.

Let L denote the quotient Lie group N/K, and let $\mathfrak{l}$ denote its Lie algebra. By Lemma 4.4.5, we know the projection $\mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^*$ dual to the inclusion $\mathfrak{n} \hookrightarrow \mathfrak{g}$ induces an isomorphism

$$\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k}) \cap (\mathfrak{g}^*)^K \cong \mathrm{Ann}_{\mathfrak{n}^*}(\mathfrak{k}) \cong \mathfrak{l}^*.$$

Let $\mu' \colon Z_K^x \to \mathfrak{l}^*$ and $\alpha' \colon \mathsf{T} Z_K^x \to \mathfrak{l}^*$ be the compositions of this isomorphism with the restrictions of μ and α, respectively, and note that

$$Z_K^x \cap \mu^{-1}(0) = Z_K^x \cap (\mu')^{-1}(0).$$

Because Z_K^x is fixed point-wise by K, the action of N on Z_K^x induces an action of the quotient $L = N/K$ on Z_K^x. We now verify that this action is twisted generalized Hamiltonian with moment map μ' and moment one-form α'.

Since μ, α, and the projection $\mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^*$ are N-equivariant, and Z_K^x consists of K-fixed points, we know that μ' and α' are L-equivariant. Now we check that μ' and α' satisfy the generalized moment map conditions for the L-action on Z_K^x. Because K fixes the points of Z_K^x, the infinitesimal action of $\mathfrak{k}$ on Z_K^x is zero, so for all $\xi \in \mathfrak{n}$ we have

$$[\xi]_{Z_K^x} = \xi_{Z_K^x},$$

where $[\xi]$ denotes the image of ξ under the quotient projection $\mathfrak{n} \twoheadrightarrow \mathfrak{n}/\mathfrak{k} \cong \mathfrak{l}$. As noted above, $\mu^\eta = 0$ and $\alpha^\eta = 0$ for all $\eta \in \mathfrak{k}$, so $(\mu')^{[\xi]} = \mu^\xi$ and $(\alpha')^{[\xi]} = \alpha^\xi$ for all $\xi \in \mathfrak{n}$. By Theorem 4.3.5 and Lemma 4.3.11, the compositions

$$Z_K^x \xrightarrow{\mu} \mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^* , \qquad \mathsf{T} Z_K^x \xrightarrow{\alpha} \mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^*$$

are a generalized moment map and moment one-form for the N-action on the (j^*H)-twisted GC manifold Z_K^x, we conclude that

$$[\xi]_{Z_K^x} = \xi_{Z_K^x} = -\mathcal{J}'\left(\mathrm{d}\mu^\xi\right) - \alpha^\xi = -\mathcal{J}'\left(\mathrm{d}(\mu')^{[\xi]}\right) - (\alpha')^\xi$$

and

$$\iota_{[\xi]_{Z_K^x}}(j^*H) = \iota_{\xi_{Z_K^x}}(j^*H) = \mathrm{d}\alpha^\xi = \mathrm{d}(\alpha')^{[\xi]}$$

for all $[\xi] \in \mathfrak{l}$.

By part (e) of Proposition 4.2.6, we know that N/K acts freely on Z_K^x, and hence also on $(\mu')^{-1}(0)$. Since $H + \alpha$ is G-equivariantly closed, we know H is closed. Using this fact, our computations from the previous paragraph, and part (b) of Remark 4.3.3, we compute

$$\begin{aligned}
\mathrm{d}_L(j^*H + \alpha')([\xi]) &= \mathrm{d}(j^*H) - \iota_{[\xi]_{Z_K^x}}(j^*H) + \mathrm{d}(\alpha')^{[\xi]} - \iota_{[\xi]_{Z_K^x}}(\alpha')^{[\xi]} \\
&= j^*(\mathrm{d}H) - \iota_{\xi_{Z_K^x}}(j^*H) + \mathrm{d}\alpha^\xi - \iota_{\xi_{Z_K^x}}(\alpha^\xi) \\
&= 0 - \mathrm{d}\alpha^\xi + \mathrm{d}\alpha^\xi - 0 \\
&= 0
\end{aligned}$$

for all $[\xi] \in (\mathfrak{n}/\mathfrak{k})^* \cong \mathfrak{l}^*$. Hence $j^*H + \alpha'$ is L-equivariantly closed. Therefore we can apply Lin–Tolman's twisted generalized reduction, Theorem 4.3.8 above, and obtain a GC structure on the quotient space

$$(\mu')^{-1}(0) \Big/ (N/K) \cong (\mu')^{-1}(0) \Big/ N \cong \left(Z_K^x \cap \mu^{-1}(0)\right) \Big/ N.$$

Recall that this structure is only natural up to B-transform. It follows that each component of $(Z_K^x \cap \mu^{-1}(0)) \Big/ N$ is a twisted GC manifold.

By varying the point $x \in M_K$, and thus varying the manifold Z_K^x, we can conclude that every component $(M_K \cap \mu^{-1}(0)) \Big/ N$ is a twisted GC manifold.

By parts (d) and (f) of Proposition 4.2.6, we know that $G \cdot M_K = M_{(K)}$ and that the inclusion $M_K \hookrightarrow M_{(K)}$ induces a homeomorphism $M_K/N \approx M_{(K)}/G$. Together with the fact that $\mu^{-1}(0)$ is G-stable, this first fact implies that $G \cdot (M_K \cap \mu^{-1}(0)) = M_{(K)} \cap \mu^{-1}(0)$. Together with the second fact, this implies that

$$(M_0)_{(K)} := \left(M_{(K)} \cap \mu^{-1}(0)\right) \Big/ G \approx \left(M_K \cap \mu^{-1}(0)\right) \Big/ N,$$

and so each component of $(M_0)_{(K)}$ inherits a twisted GC structure, natural up to B-transform.

The general case, where the reduction is taken at an arbitrary level $a \in \mathfrak{g}^*$ now follows from the shifting trick, as explained following Example 4.3.12 above. Let $\mathcal{O}_a$ be the coadjoint orbit of G through a, let ω_a be the canonical symplectic form on $\mathcal{O}_a$, and let $\mathcal{J}_a$ be the GC structure on $\mathcal{O}_a$ corresponding to the *negative* of the canonical symplectic structure, $-\omega_a$. Then the action of G on $(\mathcal{O}_a, \mathcal{J}_a)$ is generalized Hamiltonian with moment map $\mathcal{O}_a \to \mathfrak{g}^*$, $\lambda \mapsto -\lambda$, and the diagonal action of G on $(M \times \mathcal{O}_a, \mathcal{J} \oplus \mathcal{J}_a)$ is generalized Hamiltonian with moment map

$$\mu' \colon M \times \mathcal{O}_a \to \mathfrak{g}^*, \quad (x, \lambda) \mapsto \mu(x) - \lambda.$$

The inverse images $\mu^{-1}(a)$ and $(\mu')^{-1}(0)$ are G-equivariantly homeomorphic, which means their quotients are homeomorphic:

$$M_a := \mu^{-1}(a)/G \approx (\mu')^{-1}(0)/G =: (M \times \mathcal{O}_a)_0.$$

Furthermore, because this homeomorphism comes from a G-equivariant homeomorphism, we know that the orbit type partitions of these quotient spaces is preserved by the homeomorphism. Therefore, the desired result about the orbit type partition of M_a follows from the results we have already proved for the case of reduction at level zero.

Now we consider the untwisted case. Since an untwisted Hamiltonian GC manifold is simply a twisted Hamiltonian GC manifold with $H = 0$ and $\alpha = 0$, the only real difference between parts (a) and (b) of this theorem is that in part (a) we do not assume that M is compact.

Note that the only time above where we used the fact that M is compact was when showing that μ^ξ and α^ξ both vanish on Z_K^x for all $\xi \in \mathfrak{k}$, and hence that $\mu(Z_K^x)$ and $\alpha(\mathsf{T} Z_K^x)$ lie in $\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k})$.

For this non-compact case, note that since Z_K^x contains only K-fixed points, we have $\xi_{Z_K^x} = 0$ for all $\xi \in \mathfrak{k}$, so

$$\mathrm{d}\mu^\xi = \mathcal{J}'(\xi_{Z_K^x}) = \mathcal{J}'(0) = 0$$

and hence μ^ξ is locally constant on Z_K^x for all $\xi \in \mathfrak{k}$. Because each component of Z_K^x has nonempty intersection with $\mu^{-1}(0)$, it follows that $\mu^\xi = 0$ for all $\xi \in \mathfrak{k}$, so $\mu(Z_K^x) \subset \mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{k})$.

This completes the proof of (a). $\square$

Corollary 4.4.7 (Singular generalized Kähler reduction). *The results of Theorem 4.4.6 hold if all GC and twisted GC structures are replaced by GK and twisted GK structures, respectively.*

Proof. Suppose $(M, \mathcal{J}_1, \mathcal{J}_2)$ is a GK manifold, twisted or untwisted. Because a generalized Hamiltonian action on the GK manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ is simply a generalized Hamiltonian action on the GC manifold $(M, \mathcal{J}_1)$ which also preserves the structure $\mathcal{J}_2$, it is easy to check that the proof of Theorem 4.4.6 holds in precisely the same way for our present situation. We will simply note that, for any Lie subgroup K of G, because both $\mathcal{J}_1$ and $\mathcal{J}_2$ are preserved by K, by Proposition 4.1.45 we know that each component of M^K is a split submanifold of M with respect to both GC structures, so it is a GK manifold. Everything else is entirely straightforward to check. $\square$

BIBLIOGRAPHY

[Ada] Jeffrey Adams. Parameters for representations of real groups. http://atlas.math.umd.edu/papers/summer05/parameters.pdf. notes from the AIM workshop, July 2005.

[Ati82] Michael Francis Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14(1):1–15, 1982.

[BBB04] Oren Ben-Bassat and Mitya Boyarchenko. Submanifolds of generalized complex manifolds. *J. Symplectic Geom.*, 2(3):309–355, 2004.

[BCG07] Henrique Bursztyn, Gil R. Cavalcanti, and Marco Gualtieri. Reduction of Courant algebroids and generalized complex structures. *Adv. Math.*, 211(2):726–765, 2007.

[BL08] Thomas Baird and Yi Lin. Topology of generalized complex quotients, 2008. 33 pages, Preprint, arXiv:math.DG/0802.1341.

[Bri87] Michel Brion. Sur l'image de l'application moment. In *Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986)*, volume 1296 of *Lecture Notes in Math.*, pages 177–192. Springer, Berlin, 1987.

[CdS01] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, 2001.

[Cou90] Theodore James Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319(2):631–661, 1990.

[CŚ01] Richard Cushman and Jędrzej Śniatycki. Differential structure of orbit spaces. *Canad. J. Math.*, 53(4):715–755, 2001.

[DK00] J.J. Duistermaat and J.A.C. Kolk. *Lie Groups (Universitext)*. Springer, 2000.

[Dui83] J. J. Duistermaat. Convexity and tightness for restrictions of Hamiltonian functions to fixed point sets of an antisymplectic involution. *Trans. Amer. Math. Soc.*, 275(1):417–429, 1983.

[Gol09a] Timothy E. Goldberg. A convexity theorem for the real part of a Borel invariant subvariety. *Proc. Amer. Math. Soc.*, 137(4):1447–1458, 2009.

[Gol09b] Timothy E. Goldberg. Singular reduction of generalized complex manifolds, 2009. 26 pages, Preprint, arXiv:math.DG/1003.1773.

[GS82] Victor Guillemin and Shlomo Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67(3):491–513, 1982.

[GS99] Victor W. Guillemin and Shlomo Sternberg. *Supersymmetry and equivariant de Rham theory*. Mathematics Past and Present. Springer-Verlag, Berlin, 1999. With an appendix containing two reprints by Henri Cartan [MR0042426 (13,107e); MR0042427 (13,107f)].

[GS06] Victor Guillemin and Reyer Sjamaar. Convexity theorems for varieties invariant under a Borel subgroup. *Pure Appl. Math. Q.*, 2(3, part 1):637–653, 2006.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[Gua03] Marco Gualtieri. *Generalized complex structures*. PhD thesis, University of Oxford, 2003.

[Hit03] Nigel Hitchin. Generalized Calabi-Yau manifolds. *Q. J. Math.*, 54(3):281–308, 2003.

[Hu09] Shengda Hu. Hamiltonian symmetries and reduction in generalized geometry. *Houston J. Math.*, 35(3):787–811, 2009.

[JRS09] Madeleine Jotz, Tudor S. Ratiu, and Jedrzez Śniatycki. Singular reduction of dirac structures, 2009. arXiv:math.DG/0901.3062.

[Kir84] Frances Kirwan. Convexity properties of the moment mapping. III. *Invent. Math.*, 77(3):547–552, 1984.

[Kos70] Bertram Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[LT06] Yi Lin and Susan Tolman. Symmetries in generalized Kähler geometry. *Comm. Math. Phys.*, 268(1):199–222, 2006.

[Mum99] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[MW74] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.*, 5(1):121–130, 1974.

[Nit09] Yasufumi Nitta. Convexity properties of generalized moment maps, 2009. 25 pages, to appear in J. Math. Soc. Japan., arXiv:math.DG/0901.0361.

[OR04] Juan-Pablo Ortega and Tudor S. Ratiu. *Momentum maps and Hamiltonian reduction*, volume 222 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2004.

[OS00] Luis O'Shea and Reyer Sjamaar. Moment maps and Riemannian symmetric pairs. *Math. Ann.*, 317, 2000.

[SL91] Reyer Sjamaar and Eugene Lerman. Stratified symplectic spaces and reduction. *Ann. of Math. (2)*, 134(2):375–422, 1991.

[SX08] Mathieu Stiénon and Ping Xu. Reduction of generalized complex structures. *J. Geom. Phys.*, 58(1):105–121, 2008.

[Vai07] Izu Vaisman. Reduction and submanifolds of generalized complex manifolds. *Differential Geom. Appl.*, 25(2):147–166, 2007.

[Vai10] Izu Vaisman. From generalized kähler to generalized sasakian structures, 2010. 25 pages, arXiv:math.DG/1004.0800.

[Wel08] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2008. With a new appendix by Oscar Garcia-Prada.